Optimal Control Theory and the Linear Quadratic Regulator

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CS/Stat 184: Introduction to Reinforcement Learning
Fall 2022

Today

- Feedback from last lecture
- Recap
- Finite-horizon discrete MDPs
- General optimal control problem
- The linear quadratic regulator (LQR) problem

Feedback from feedback forms

1. Thank you to everyone who filled out the forms!
2.

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Recap

- For discrete MDPs, we covered some great algorithms for computing the optimal policy (reminder, we haven't done any *learning* in MDPs yet)
- But all algorithms polynomially in the size of the state and action spaces… what if one or both are infinite?
- In this unit (next 3 lectures), we will discuss computation of good/optimal policies in continuous state and action spaces (still no learning yet!)

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Recap

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Finite Horizon MDPs

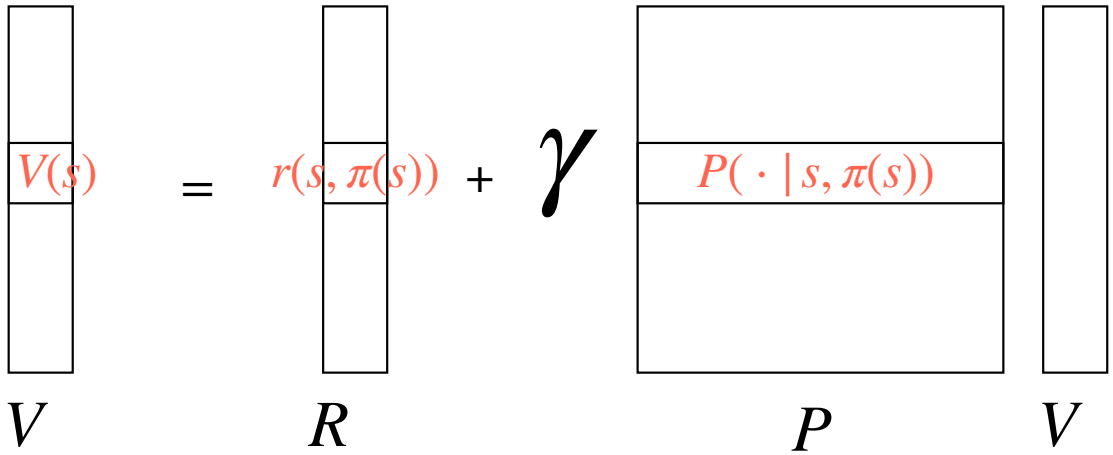
Value Iteration Algorithm:

1. Initialization: $V^0 : \|V^0\|_\infty \in \left[0, \frac{1}{1-\gamma}\right]$

2. Iterate until convergence: $V^{t+1} \leftarrow \mathcal{T} V^t$

Exact Policy Evaluation: Matrix Version

- Define: $R \in \mathbb{R}^{|S|}$, where $R_s^\pi = r(s, \pi(s))$, and $P^\pi \in \mathbb{R}^{|S| \times |S|}$, where $P^\pi_{s',s} = P(s' \mid s, \pi(s))$
- So we want to find $V \in \mathbb{R}^{|S|}$, s.t. $V = R^\pi + \gamma P^\pi V$



- **Algo:** compute $V = (I - \gamma P^\pi)^{-1} R^\pi$

  One can show that $I - \gamma P^\pi$ is full rank (thus invertible).
- **Runtime:** This approach runs in time $O(|S|^3)$.

An Iterative Version for Policy Eval

Algorithm (Iterative PE):

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2. Iterate until convergence: $V^{t+1} \leftarrow R + \gamma P V^t$

Policy Iteration (PI)

- Initialization: choose a policy $\pi^0 : S \mapsto A$
- For $t = 0,1,\ldots$
  1. **Policy Evaluation**: compute $V^{\pi^t}(s)$ and $Q^{\pi^t}(s,a)$, where
     $$Q^{\pi^t}(s,a) = r(s,a) + \gamma \sum_{s'} P(s' \mid s,a) V^{\pi^t}(s')$$
  2. **Policy Improvement**: set
     $$\pi^{t+1}(s) := \arg\max_a Q^{\pi^t}(s,a)$$

What's computation complexity per iteration?

$O(|S|^3 + |S|^2|A|)$

Recap

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Finite Horizon MDPs

Finite horizon Markov Decision Process

$$\mathscr{M} = \{S, A, r, P, H\},$$
$$r : S \times A \mapsto [0,1], H \in \mathbb{N}, P : S \times A \mapsto \Delta(S)$$

$$\mathbb{E}\left[\sum_{t=0}^{H-1} r(s_t, a_t)\right]$$

sometimes we have s_0

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$$\pi := \left\{\pi_0, \pi_1, \ldots, \pi_{H-1}\right\}, \pi_h : S \mapsto A, \forall h$$

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Policy interacts with the MDP as follows to sample a trajectory $\tau = \{s_0, a_0, r_0, s_1, a_1, r_1, \ldots, s_{H-1}, a_{H-1}, r_{H-1}\}$ as follows:

$$a_0 = \pi_0(s_0), s_1 \sim P(\cdot \mid s_0, a_0), a_1 = \pi_1(s_1), \ldots \quad s_h \sim P(\cdot \mid s_{h-1}, a_{h-1}), a_h = \pi_h(s_h) \ldots$$

V/Q functions in Finite horizon MDP

$a_h = \pi_h(s_h)$

$$V_h^\pi(s) = \mathbb{E}\left[\sum_{\tau=h}^{H-1} r(s_\tau, a_\tau) \,\middle|\, s_h = s\right]$$

$$Q_h^\pi(s,a) = \mathbb{E}\left[\sum_{\tau=h}^{H-1} r(s_\tau, a_\tau) \,\middle|\, (s_h, a_h) = (s,a)\right]$$

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Bellman Consistency Equation:

$$Q_h^\pi(s,a) = r(s,a) + \mathbb{E}_{s' \sim P(s,a)}\left[V_{h+1}^\pi(s')\right]$$

Compute Optimal Policy via Dynamic Programming

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Summary on Finite horizon MDP

$$\mathscr{M} = \{S, A, r, P, H\},$$
$$r : S \times A \mapsto [0,1], H \in \mathbb{N}, P : S \times A \mapsto \Delta(S)$$

Comparing to the infinite horizon, discounted MDP:

1. Policy will be time dependent
2. DP takes H ~~steps~~ iterations to compute $\pi^\star$
   - total computation time is $O(H|S|^2|A|)$
   - no need to use contraction argument and no discount factor
3. Extension to non-stationary setting works immediately:
   (i.e. with a non-stationary transition model: $P_0(s'|s,a), P_1(s'|s,a), \ldots P_{H-1}(s'|s,a)$)

$r_0(s,a), \ldots \quad r_{H-1}(s,a)$

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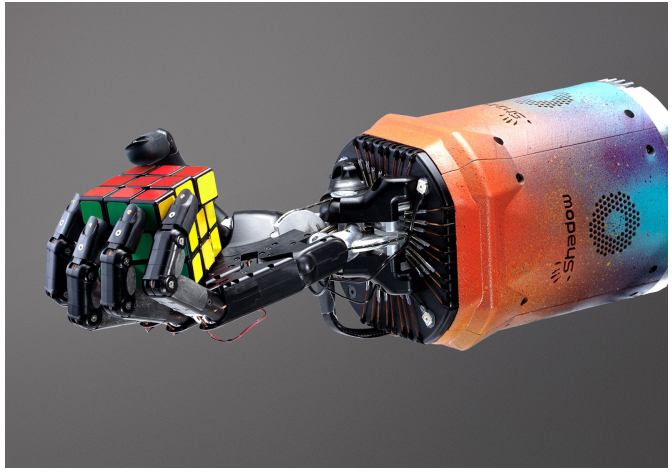
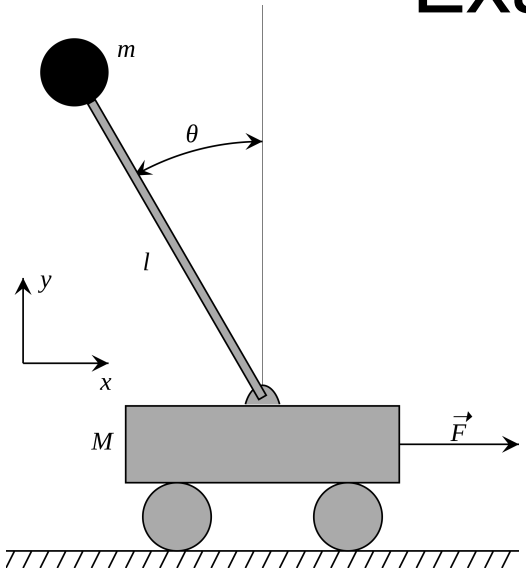
Robotics and Controls



Example: CartPole



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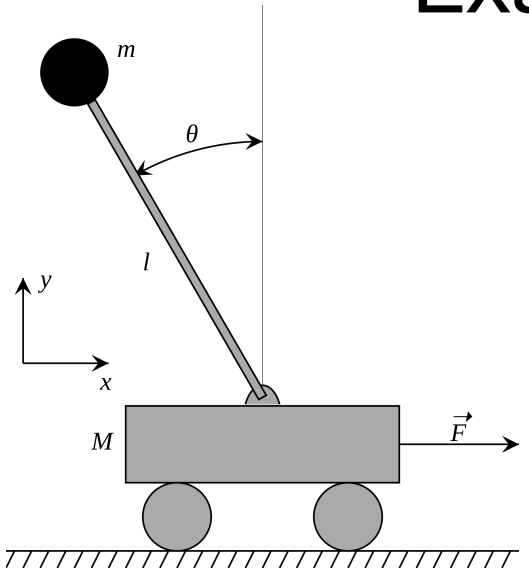

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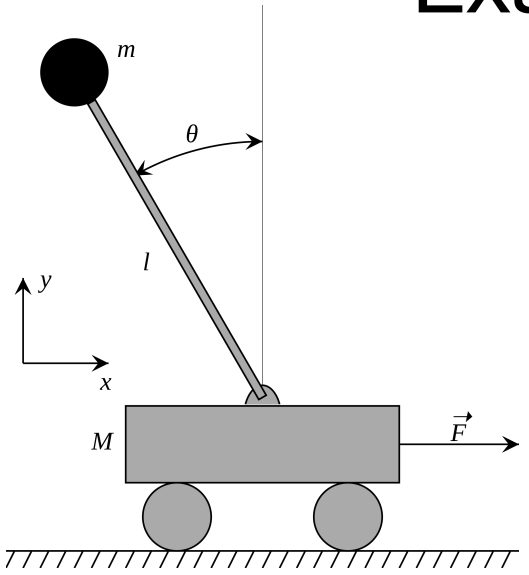

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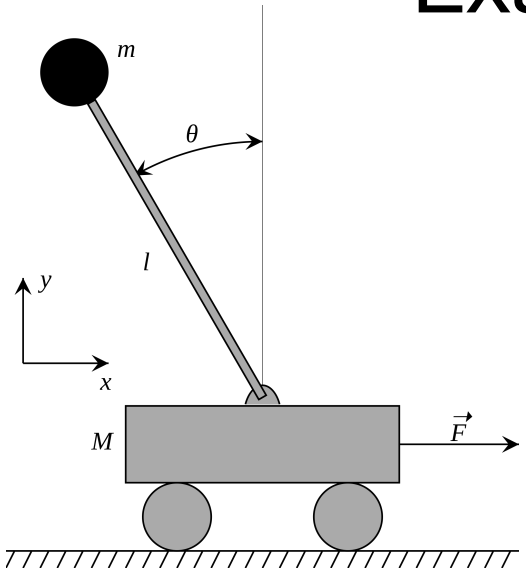

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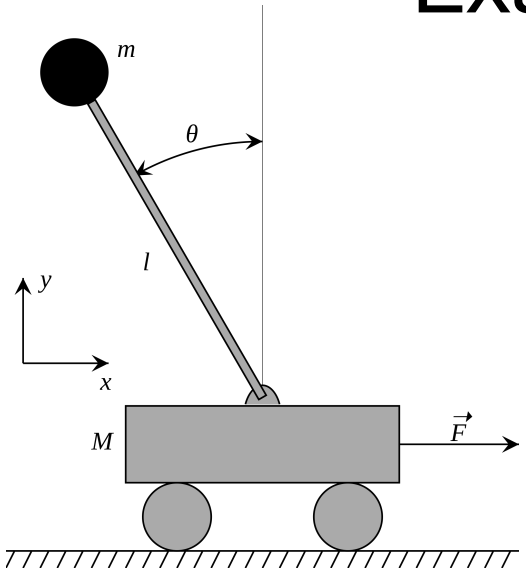

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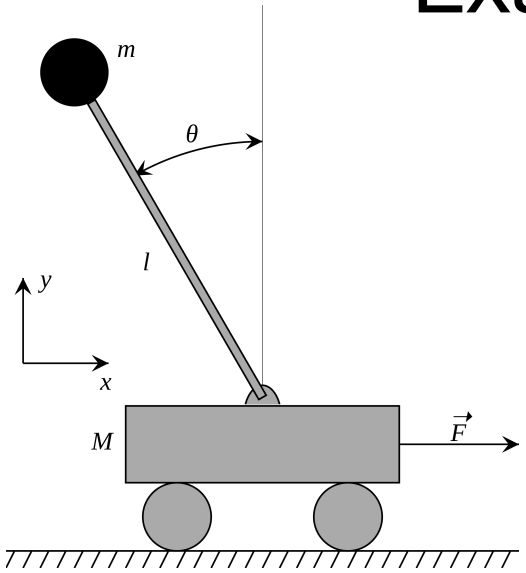

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$$\min_{\pi_0,\ldots,\pi_{T-1}:S\to A} \mathbb{E}\left[\sum_{t=0}^{T-1} c(s_t, a_t)\right] \quad \text{s.t.} \quad s_{t+1} = f(s_t, a_t),\ s_0 \sim \mu_0$$

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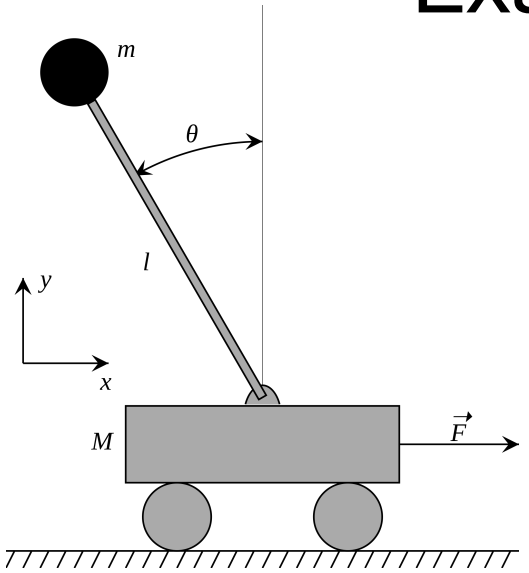

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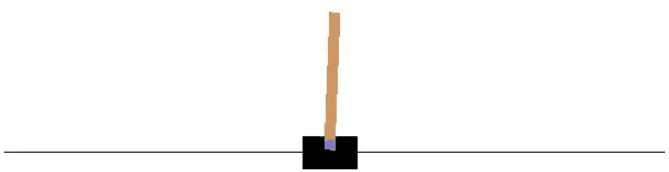

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- Note c_T separated out because by convention there is no a_T

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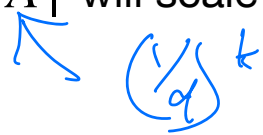

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E.g., $\epsilon = 0.01, d = k = 10$ gives $|S|^2|A|$ on the order of 10^{60}...

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E.g., $\epsilon = 0.01, d = k = 10$ gives $|S|^2|A|$ on the order of 10^{60}…

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Discretize to finite state/action spaces?

$$s \in \mathbb{R}^d, a \in \mathbb{R}^k$$

Idea: Round states and actions onto an ϵ-grid of their spaces; then use tools from finite MDPs

E.g., if $\epsilon = 0.01$, round s and a to 2 decimal places

Assuming state/control spaces are bounded, this makes both finite

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So why not rely on this more formally by assuming smoothness/structure on the dynamics f and cost c?

Today

- ✓ Feedback from last lecture
- ✓ Recap
- ✓ Finite-horizon discrete MDPs
- ✓ General optimal control problem
  - The linear quadratic regulator (LQR) problem

The Linear Quadratic Regulator (LQR)

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- Note lack of subscripts on c (except at T) and f: time-homogeneous

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In fact, because the LQR model is so well-studied in control theory, many human-engineered systems are designed to be approximately linear where possible

That said, it is indeed far too simple for many more complex (nonlinear) systems, yet in a couple lectures we will see how to extend its ideas to such systems to get surprisingly good solutions to apparently intractable nonlinear control problems

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Same trick to approximate velocity (derivative of position) via positions p_t:

$$v_t = \frac{p_t - p_{t-1}}{\delta}$$

$p_{t+1} = p_t + \delta v_t$

$v_{t+1} = v_t + \frac{\delta}{m} a_t$

$s_t = \begin{bmatrix} p_t \\ v_t \end{bmatrix}$

$s_{t+1} = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_t \\ v_t \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\delta}{m} \end{bmatrix} a_t$

A, B, s_t

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Looking ahead: we will show next lecture that the optimal control/policy is linear:

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Plugging in gives a relatively simple form for the expected state:

$$\mathbb{E}[s_t \mid s_0, a_t = -K_t s_t] = \left(\Pi_{i=0}^{t-1}(A - BK_i)\right) s_0$$

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Next time: we will solve for the optimal policy π via dynamic programming on these

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1-minute feedback form: https://bit.ly/3RHtlxy