From LQR to Nonlinear Control

Lucas Janson and Sham Kakade

CS/Stat 184: Introduction to Reinforcement Learning

Fall 2022

Today

- Feedback from last lecture
- Recap
- Locally linearization
- Iterative LQR

Feedback from feedback forms

1. Thank you to everyone who filled out the forms!
2.

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Recap: LQR

Problem Statement (finite horizon, time homogeneous):

$$\arg \min_{\pi_0,\ldots,\pi_{T-1}:\mathbb{R}^d \to \mathbb{R}^k} \mathbb{E}\left[s_T^\top Q s_T + \sum_{t=0}^{T-1} (s_t^\top Q s_t + a_t^\top R a_t) \right]$$

such that $s_{t+1} = As_t + Ba_t + w_t\,, \quad s_0 \sim \mu_0\,, \quad a_t = \pi_t(s_t)\,, \quad w_t \sim N(0,\sigma^2 I)$

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- States $s_t \in \mathbb{R}^d$
- Actions/controls $a_t \in \mathbb{R}^k$
- Additive noise $w_t \sim \mathcal{N}(0,\sigma^2 I)$
- Dynamics linear with state coefficient matrix $A \in \mathbb{R}^{d\times d}$ and action coefficient matrix $B \in \mathbb{R}^{d\times k}$
- Cost function quadratic with positive semidefinite state coefficient matrix $Q \in \mathbb{R}^{d\times d}$ and positive semidefinite action coefficient matrix $R \in \mathbb{R}^{k\times k}$

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We showed that $V_t^\star(s) = s^\top P_t s + p_t$, where:

$$P_t = Q + A^\top P_{t+1} A - A^\top P_{t+1} B (R + B^\top P_{t+1} B)^{-1} B^\top P_{t+1} A$$

$$p_t = \operatorname{tr}\left(\sigma^2 P_{t+1}\right) + p_{t+1}$$

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Optimal policy has nothing to do with initial distribution μ_0 or the noise σ^2!

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But what about problems with nonlinear dynamics and/or nonquadratic costs?

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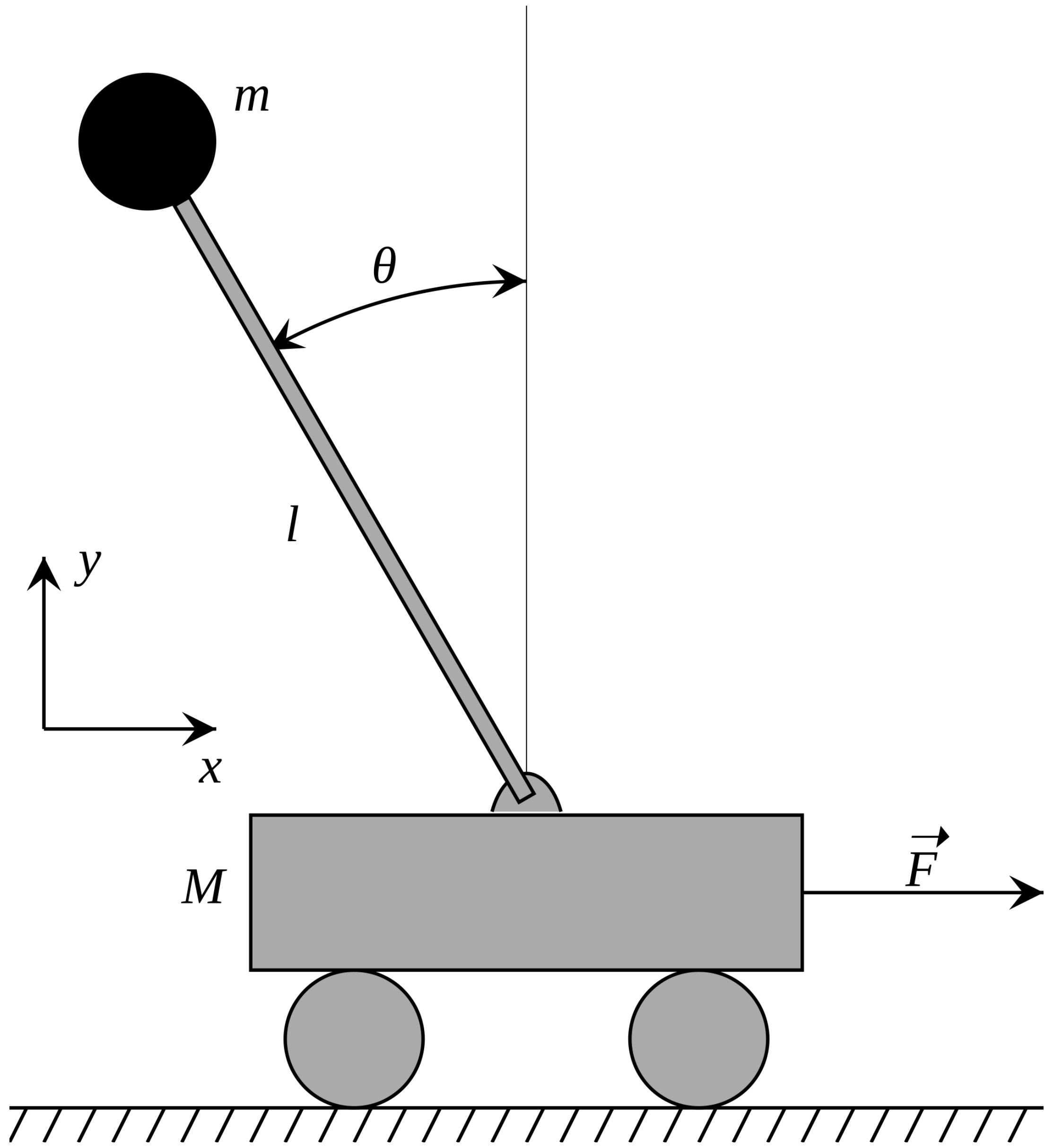

Goal: stabilizing around the goal $(s = s^\star, a = a^\star)$

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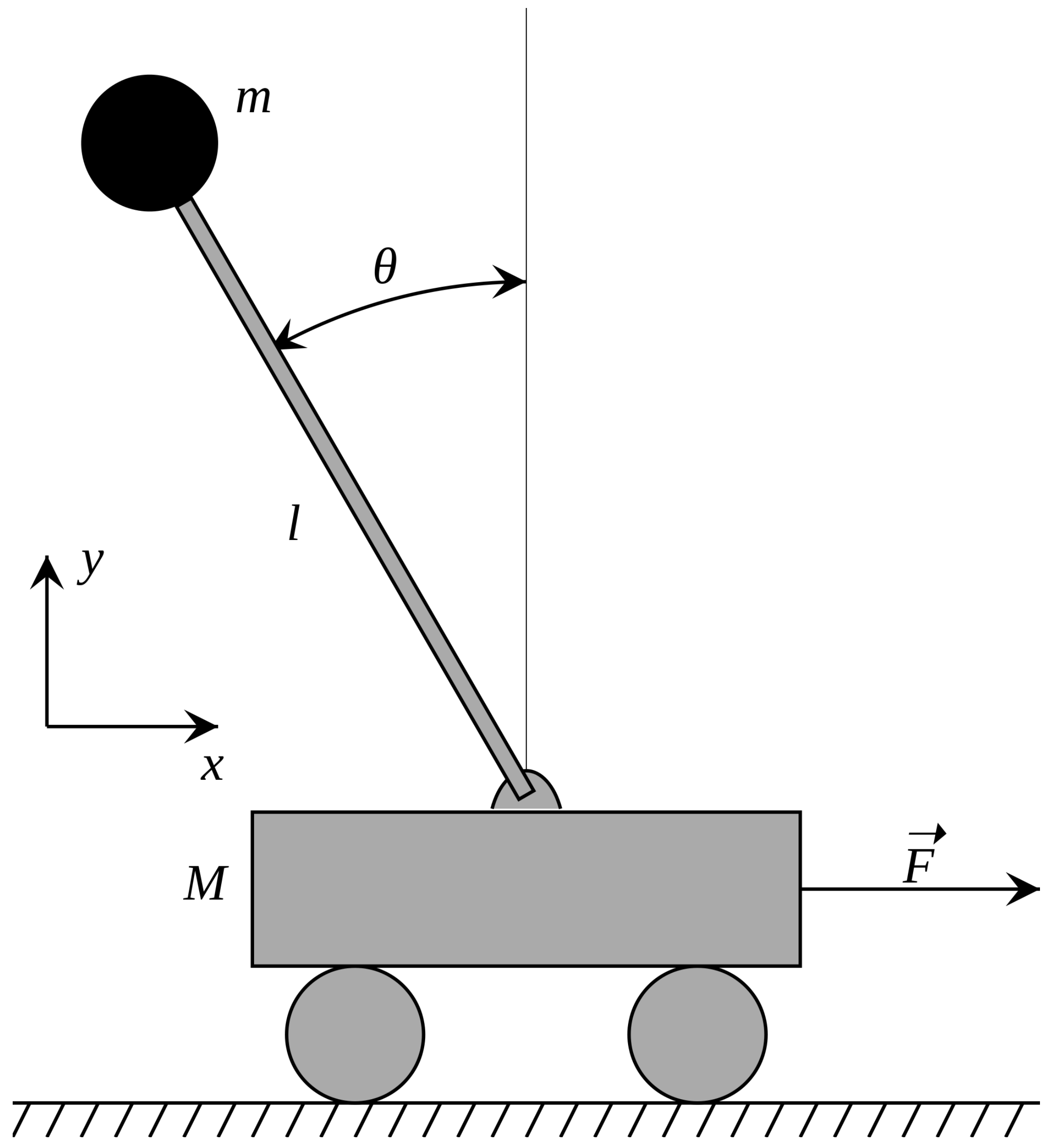

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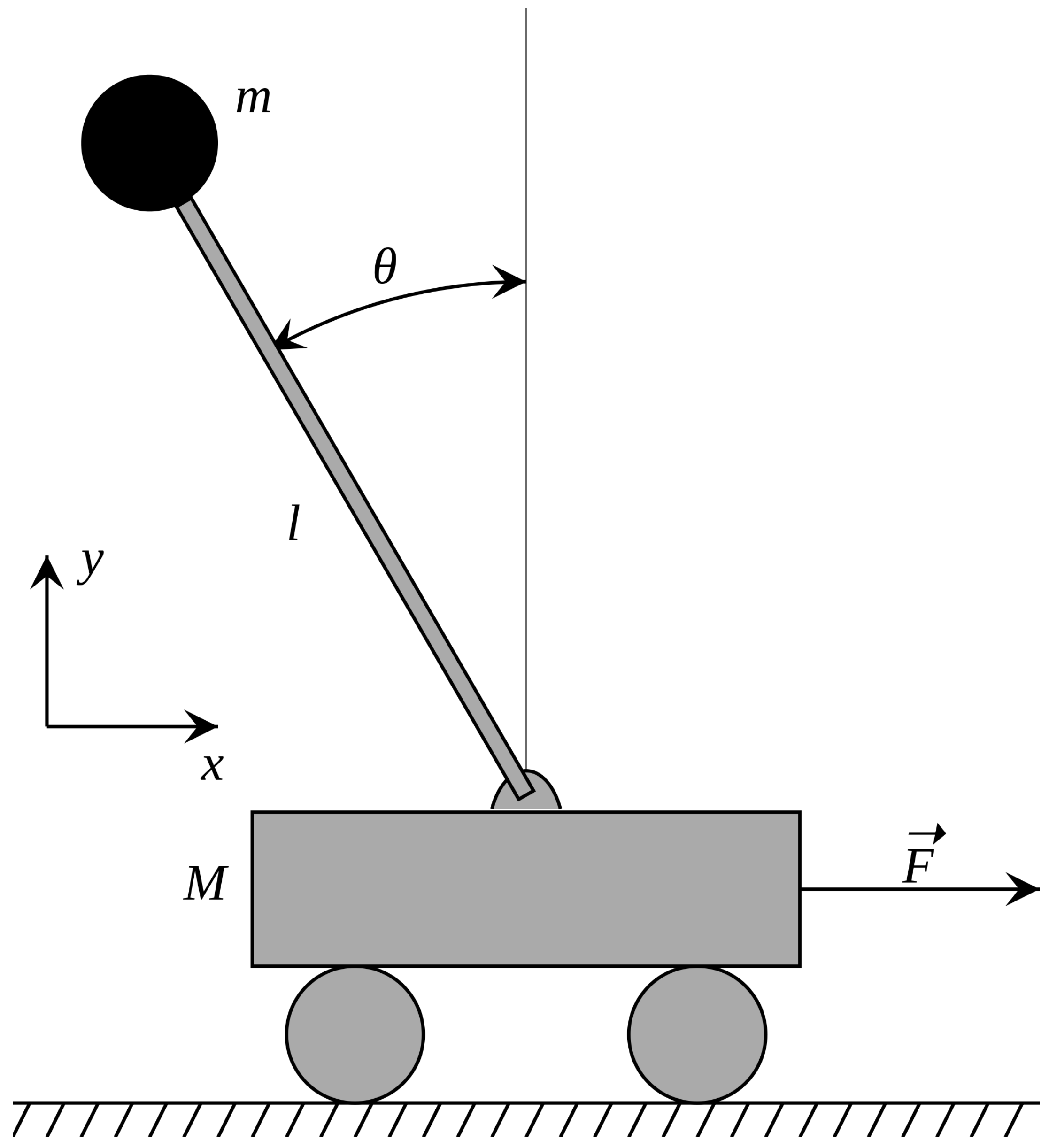

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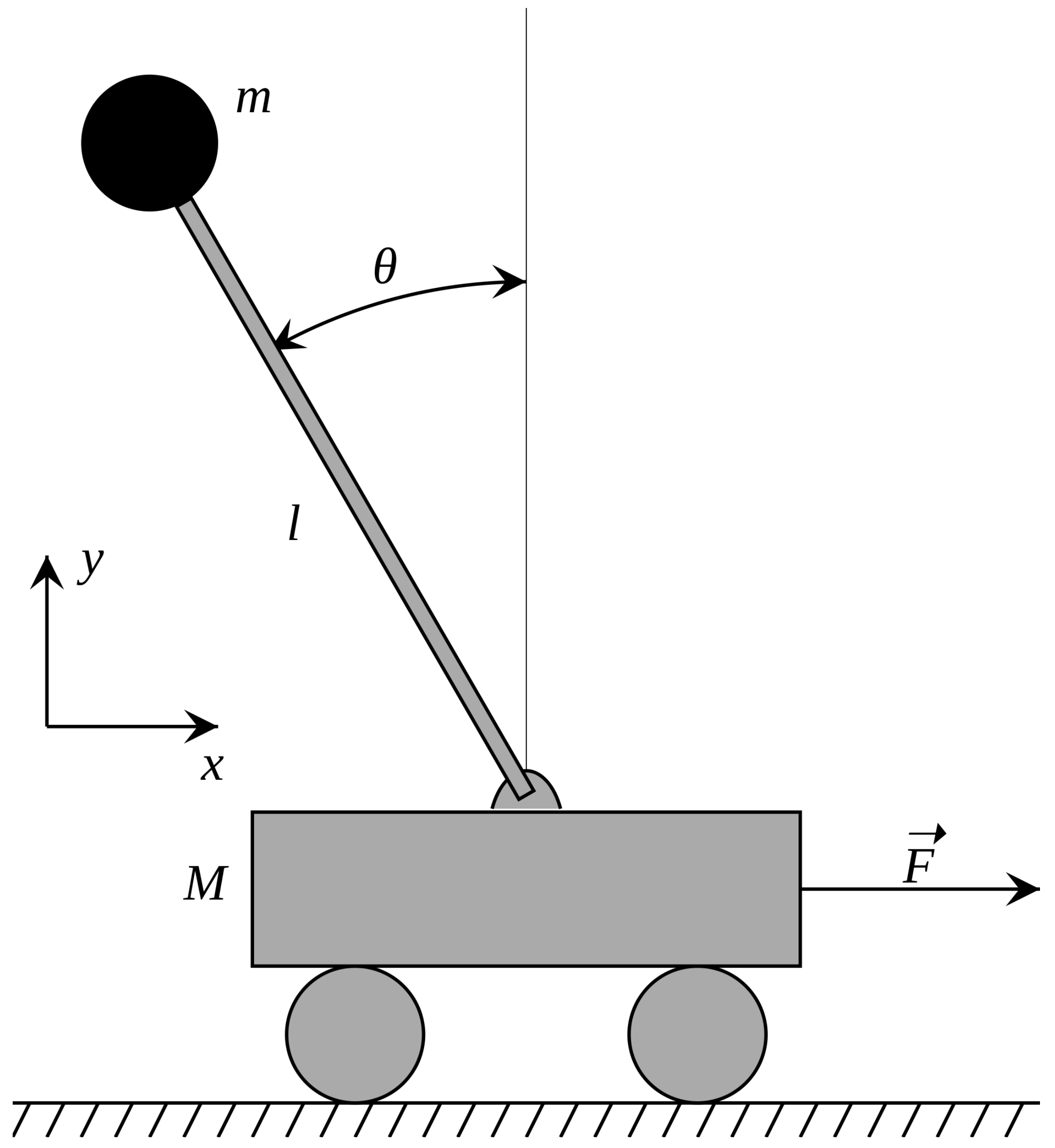

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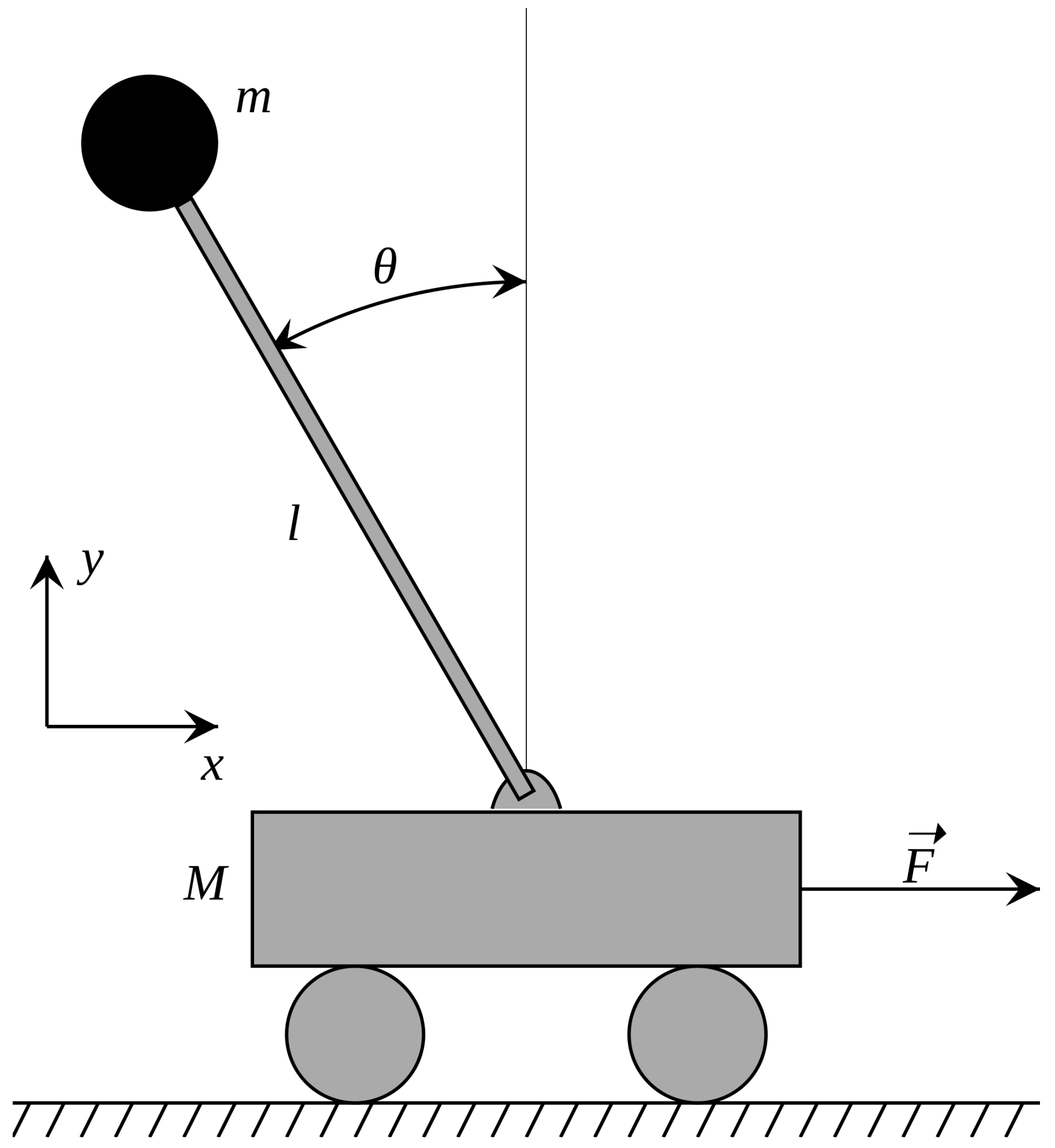

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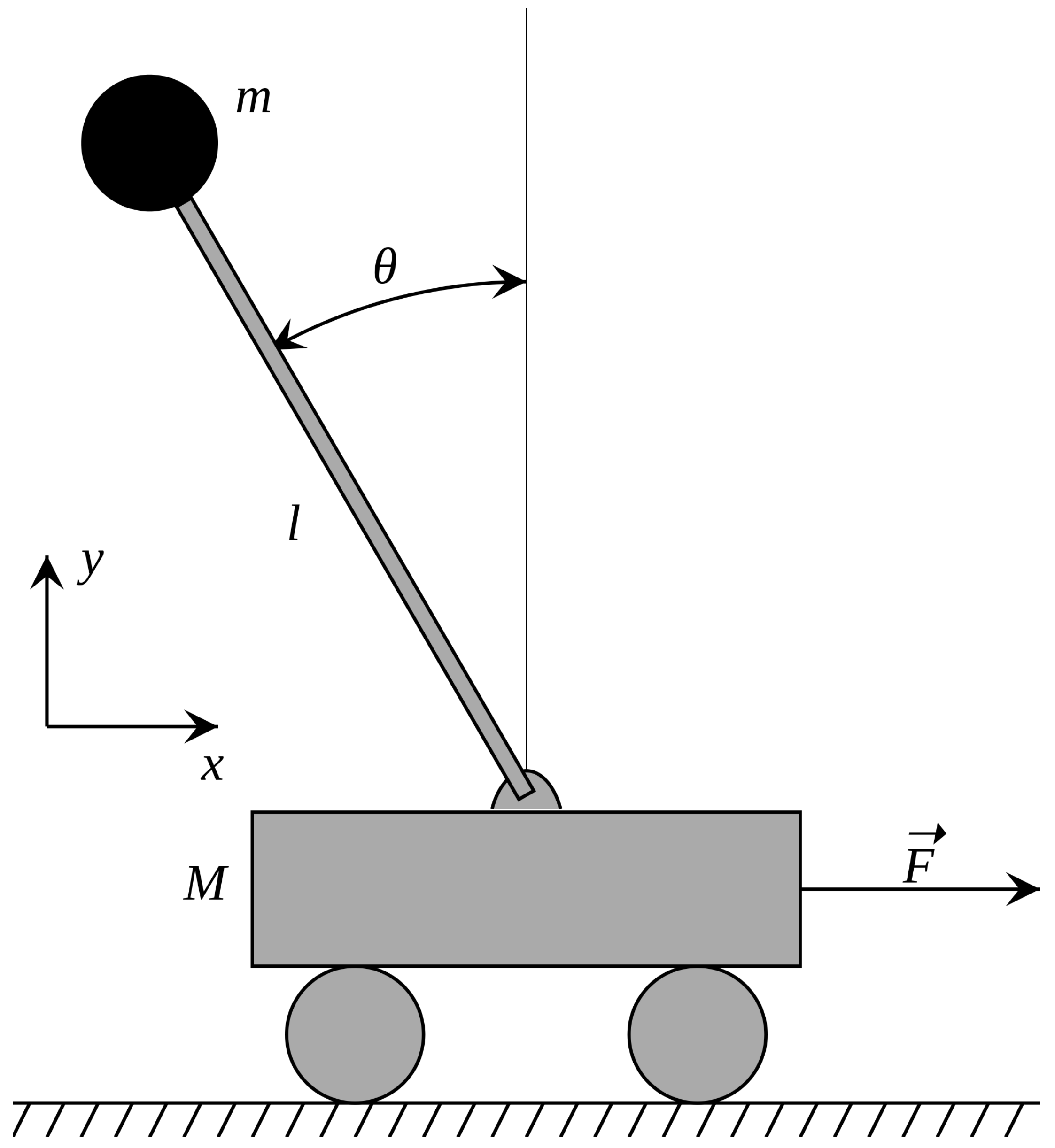

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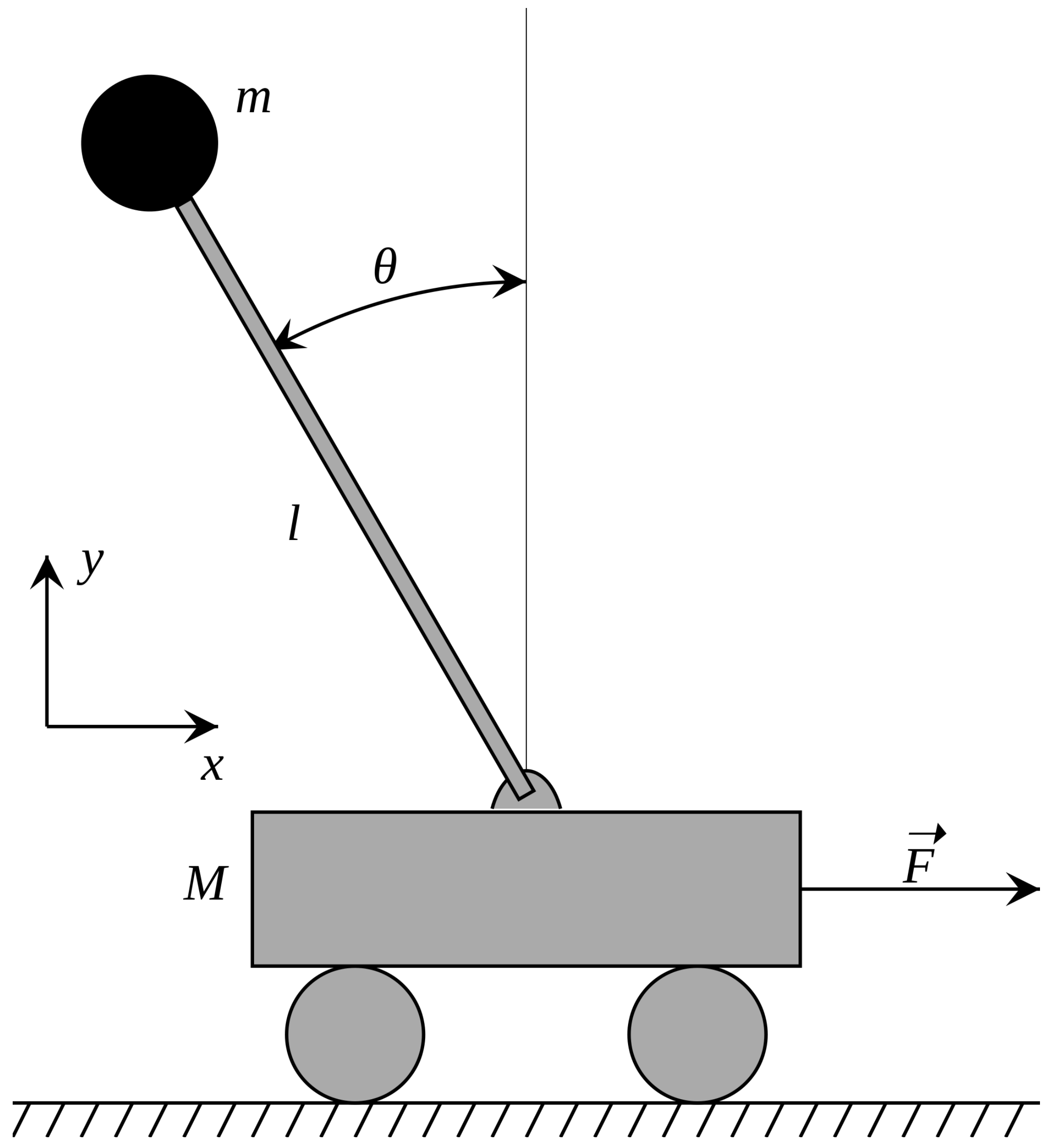

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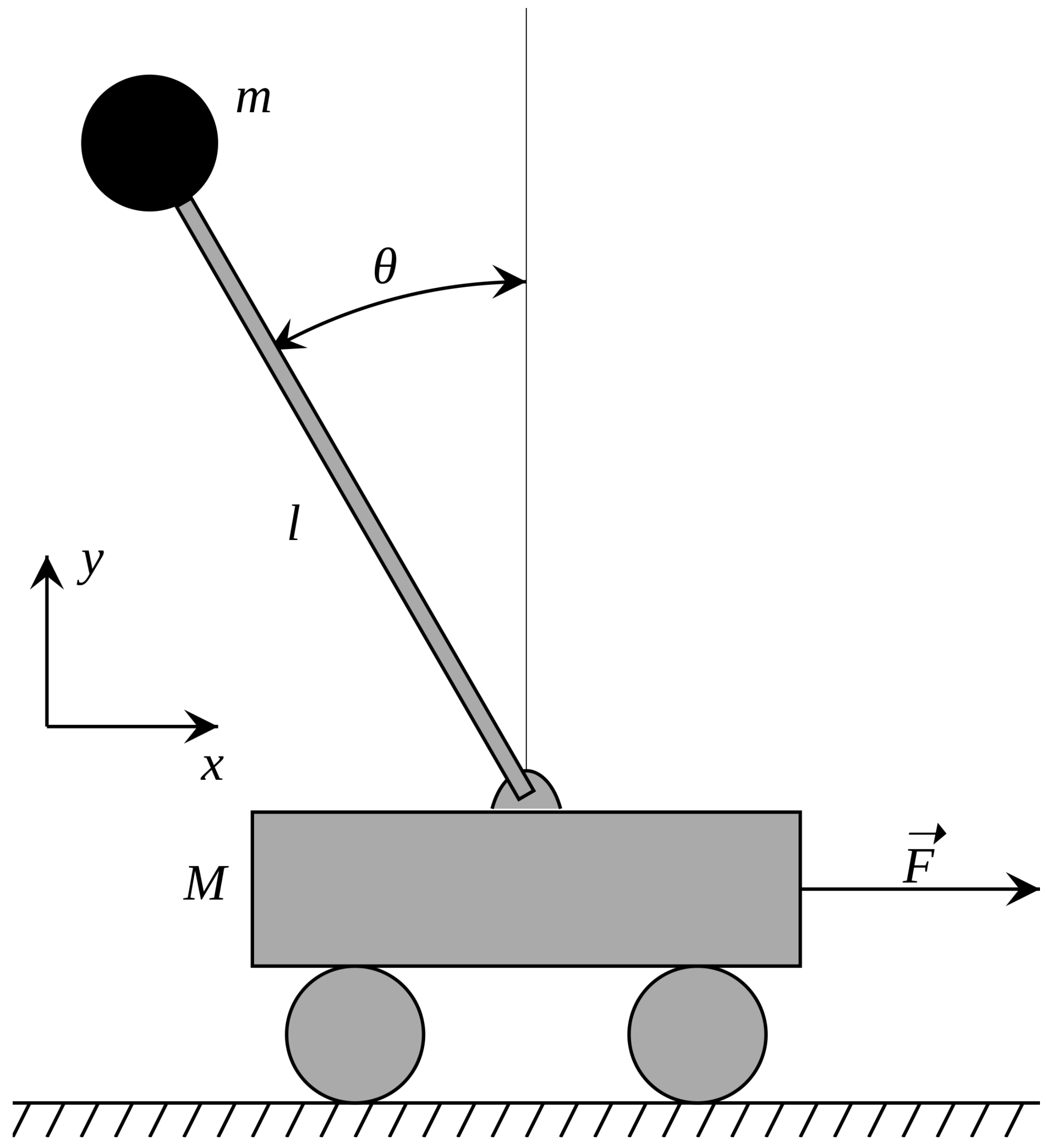

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$$\nabla_s f(s, a), \nabla_a f(s, a), \nabla_s c(s, a), \nabla_a c(s, a),$$
$$\nabla_s^2 c(s, a), \nabla_a^2 c(s, a), \nabla_{s,a}^2 c(s, a)$$

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where:

$$\nabla_s f(s,a) \in \mathbb{R}^{d\times d}, \nabla_s f(s,a)[i,j] = \frac{\partial f[i]}{\partial s[j]}(s,a), \quad \nabla_u f(s,a) \in \mathbb{R}^{d\times k}, \nabla_a f(s,a)[i,j] = \frac{\partial f[i]}{\partial a[j]}(s,a)$$

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$$\nabla_s c(s, a) \in \mathbb{R}^d, \quad \nabla_s c(s, a)[i] = \frac{\partial c}{\partial s[i]}(s, a),$$

$$\nabla_a c(s, a) \in \mathbb{R}^k, \quad \nabla_a c(s, a)[i] = \frac{\partial c}{\partial a[i]}(s, a),$$

$$\nabla_s^2 c(s, a) \in \mathbb{R}^{d \times d}, \quad \nabla_s^2 c(s, a)[i, j] = \frac{\partial^2 c}{\partial s[i] \partial s[j]}(s, a),$$

$$\nabla_{a,s}^2 c(s, a) \in \mathbb{R}^{k \times d}, \quad \nabla_{a,s}^2 c(s, a)[i, j] = \frac{\partial^2 c}{\partial a[i] \partial s[j]}(s, a)$$

Local Linearization: Putting it all Together

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Rearranging terms, we get back to the following formulation:

$$\arg \min_{\pi_0,\ldots,\pi_{T-1}:\mathbb{R}^d\to\mathbb{R}^k} \quad \mathbb{E}\left[\sum_{t=0}^{T-1}(s_t{}^\top Q s_t + a_t{}^\top R a_t + a_t{}^\top M s_t + s_t{}^\top q + a_t{}^\top r + c)\right]$$

such that $\quad s_{t+1} = As_t + Ba_t + v\,, \quad s_0 \sim \mu_0\,, \quad a_t = \pi_t(s_t)$

(HW3 problem)

Summary So far:

For tasks such as balancing near goal state $(s^\star, a^\star)$,
we can perform **first order Taylor expansion on** $f(s, a)$,
and **second order Taylor expansion on** $c(s, a)$ around the balancing point $(s^\star, a^\star)$

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Last step: checking some practical issues

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In practice, we force them to be positive definite:

Given a *symmetric matrix* $H \in \mathbb{R}^{d \times d}$,

we compute the eigen-decomposition $H = \sum_{i=1}^{d} \sigma_i u_i u_i^\top$, and we approximate H as

$$H \approx \sum_{i=1}^{d} \mathbf{1}(\sigma_i > 0)\sigma_i u_i u_i^\top + \lambda I,$$

for some small $\lambda > 0$

Computing Approximate Derivatives

Recall our assumption: we only have black-box access to f & c:

i.e., unknown analytical form, but given any (s, a), the black boxes outputs s', c, where

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First implement finite differencing procedure for $\partial c / \partial a[i]$, and then perform another finite differencing with respect to $s[j]$ on top of the first finite differencing procedure for $\partial c / \partial a[i]$

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4. The approximation is an LQR, so we know how to compute the optimal policy

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But when s_t is far away from $s^\star$ or a_t needs to be far from $a^\star$ for any t, first/second-order Taylor expansion is not accurate anymore

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After linearization and quadratization at time t around waypoint $(\bar{s}_t, \bar{a}_t)$, $\forall t$, re-arranging terms gives:

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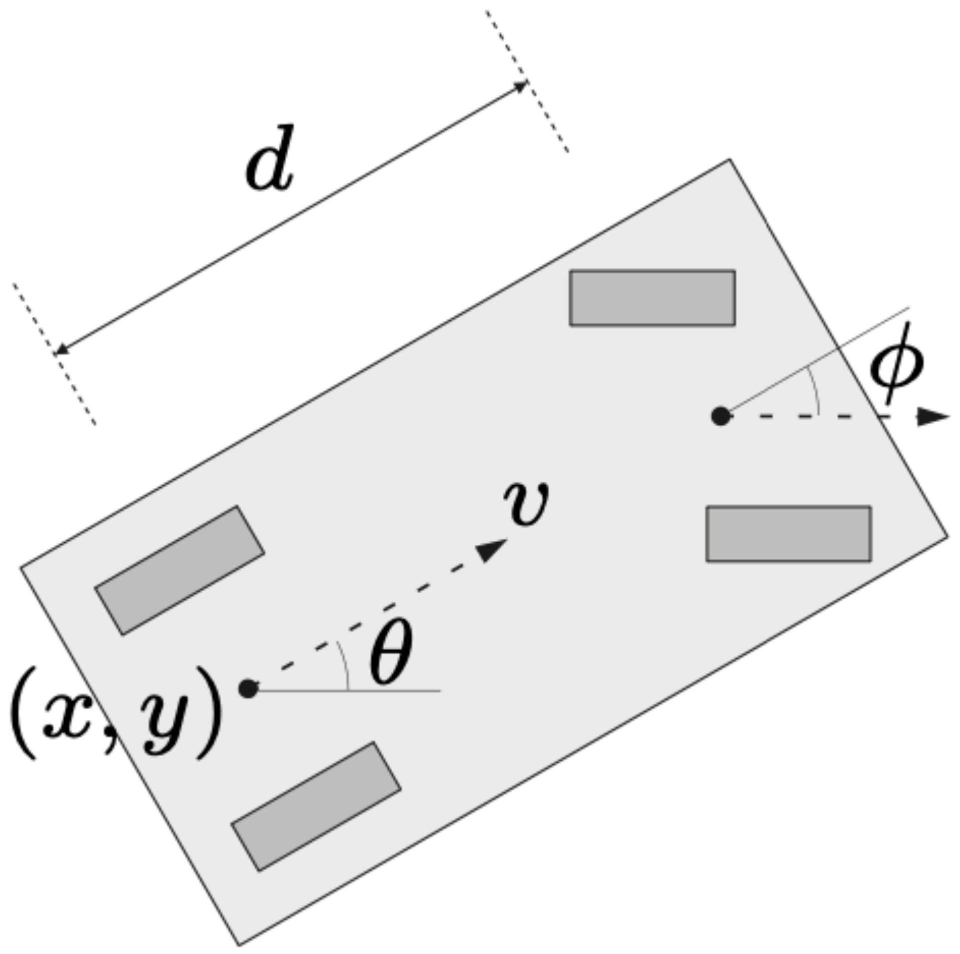
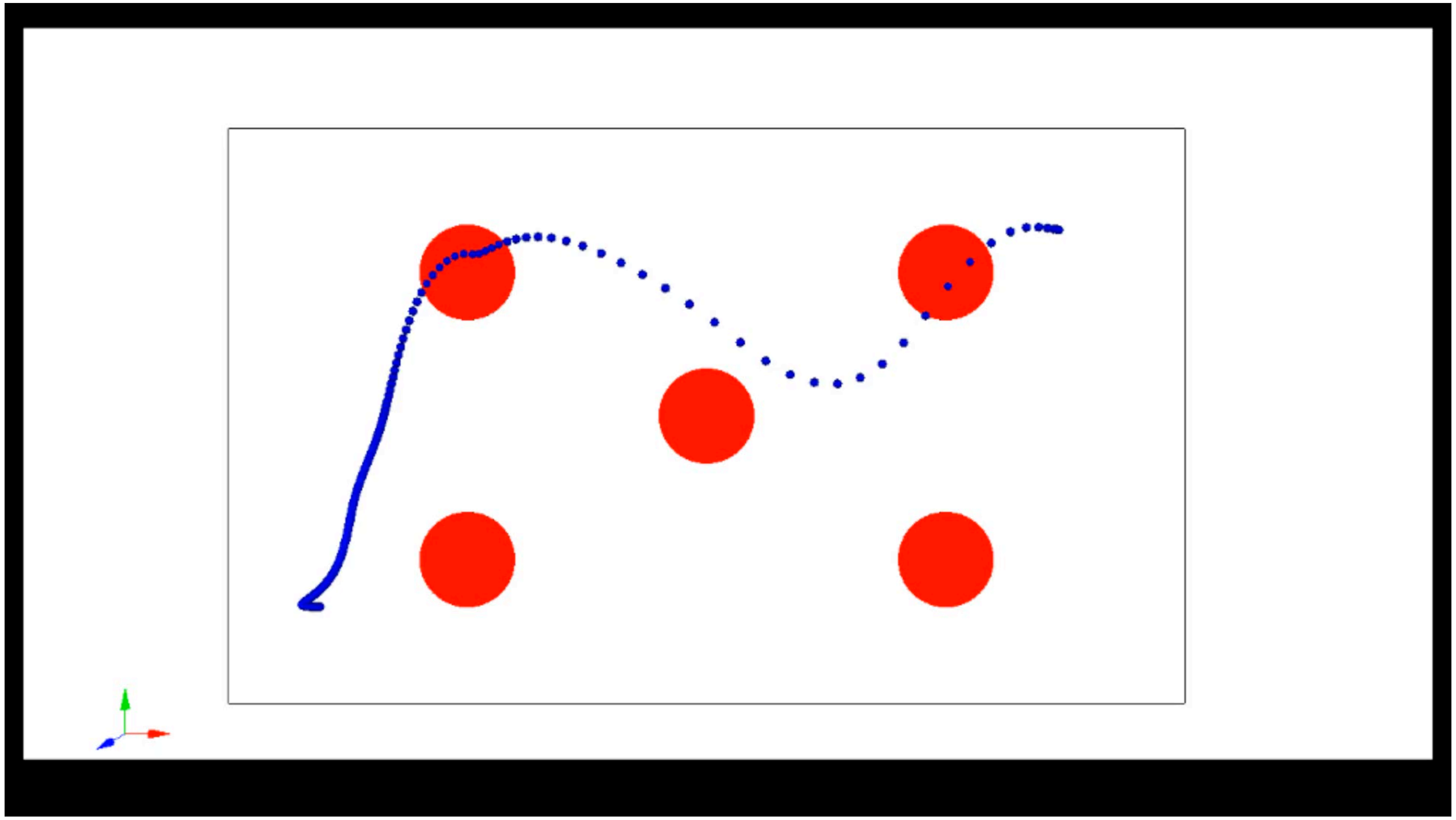
This optimization is tractable because it is 1-dimensional!

Example:

2-d car navigation

Cost function is designed such that it gets to the goal without colliding with obstacles (in red)





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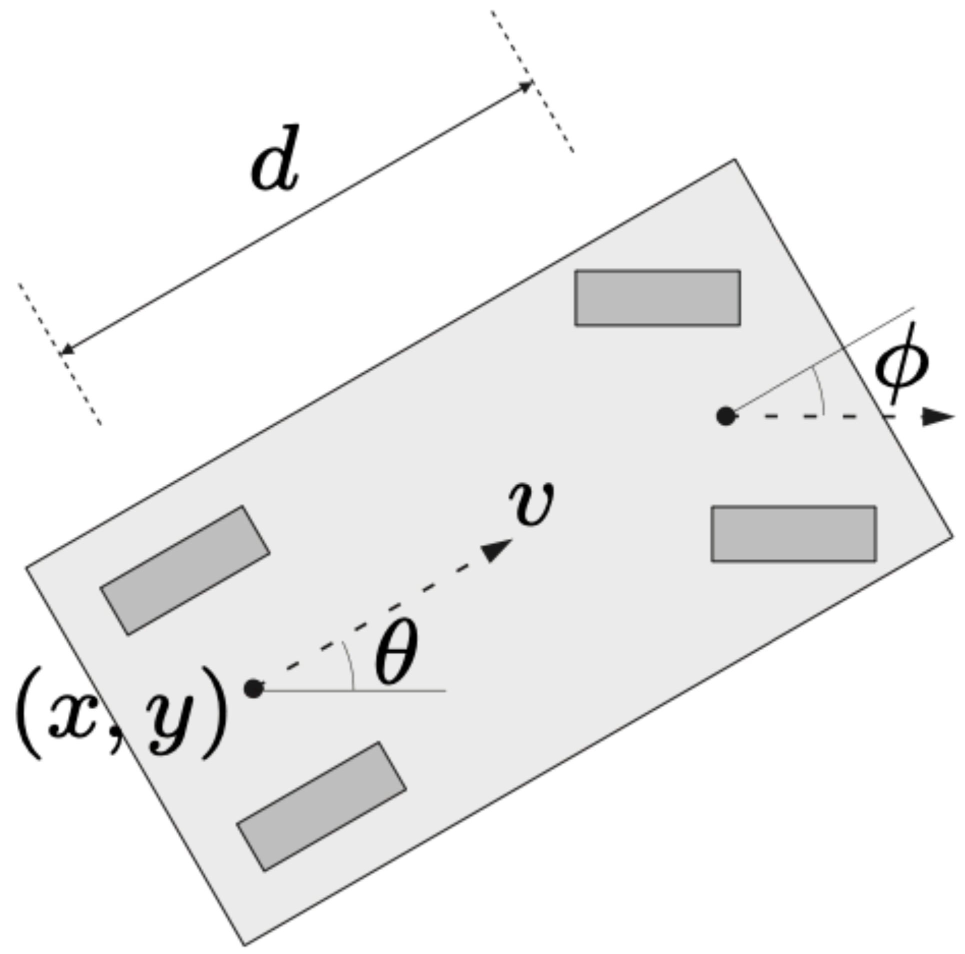

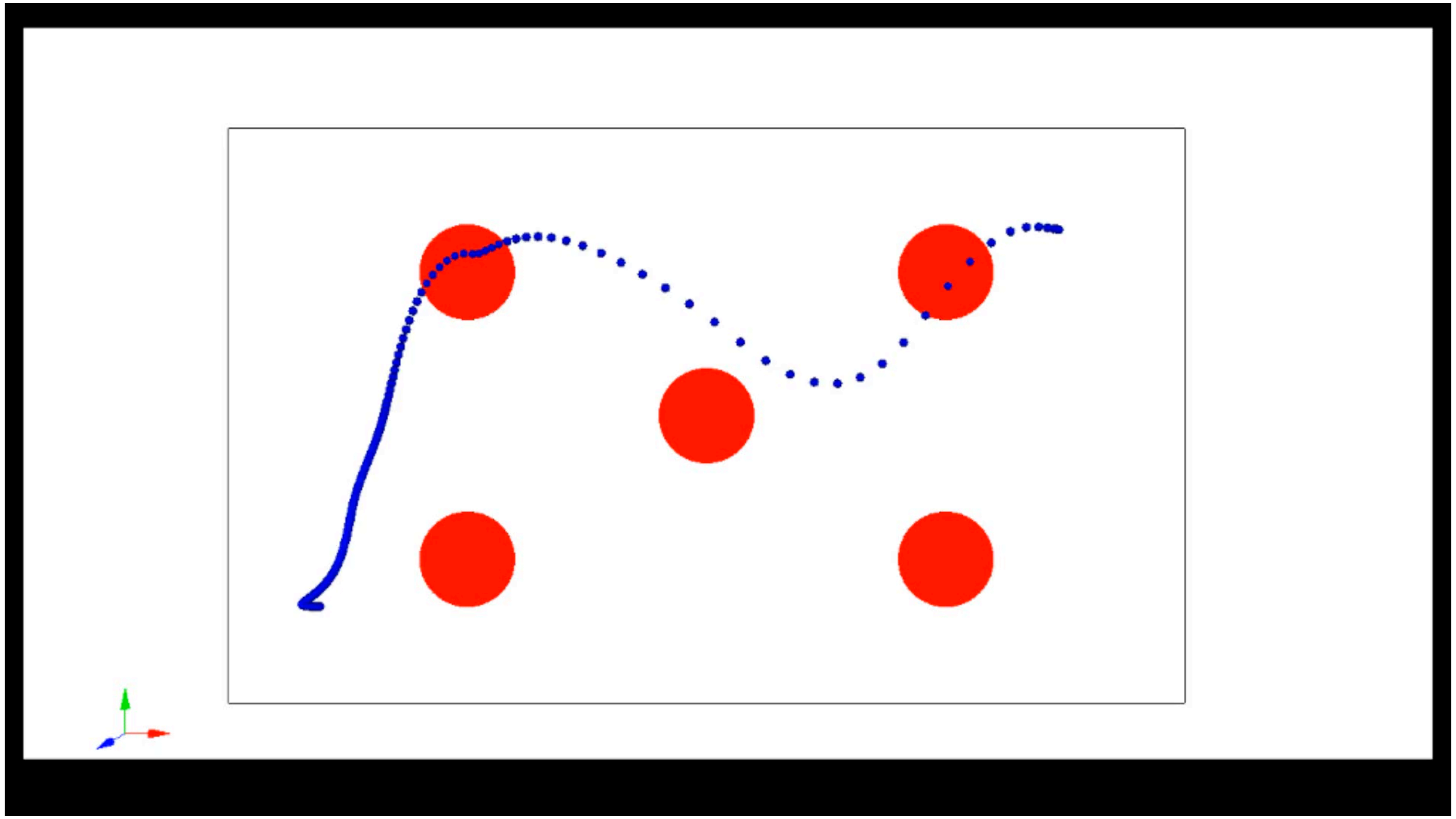

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Today

- Feedback from last lecture
- Recap
- Locally linearization
- Iterative LQR

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1-minute feedback form: https://bit.ly/3RHtlxy