Exploration: Contextual Bandits

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CS/Stat 184: Introduction to Reinforcement Learning

Fall 2022

Today

- Recap
- LinUCB algorithm for contextual bandits

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Uniform confidence bounds via Hoeffding + Union Bound

$$\mathbb{P}\left(\forall k \leq K, t < T, |\hat{\mu}_t^{(k)} - \mu^{(k)}| \leq \sqrt{\ln(2TK/\delta)/2N_t^{(k)}} \right) \geq 1 - \delta$$

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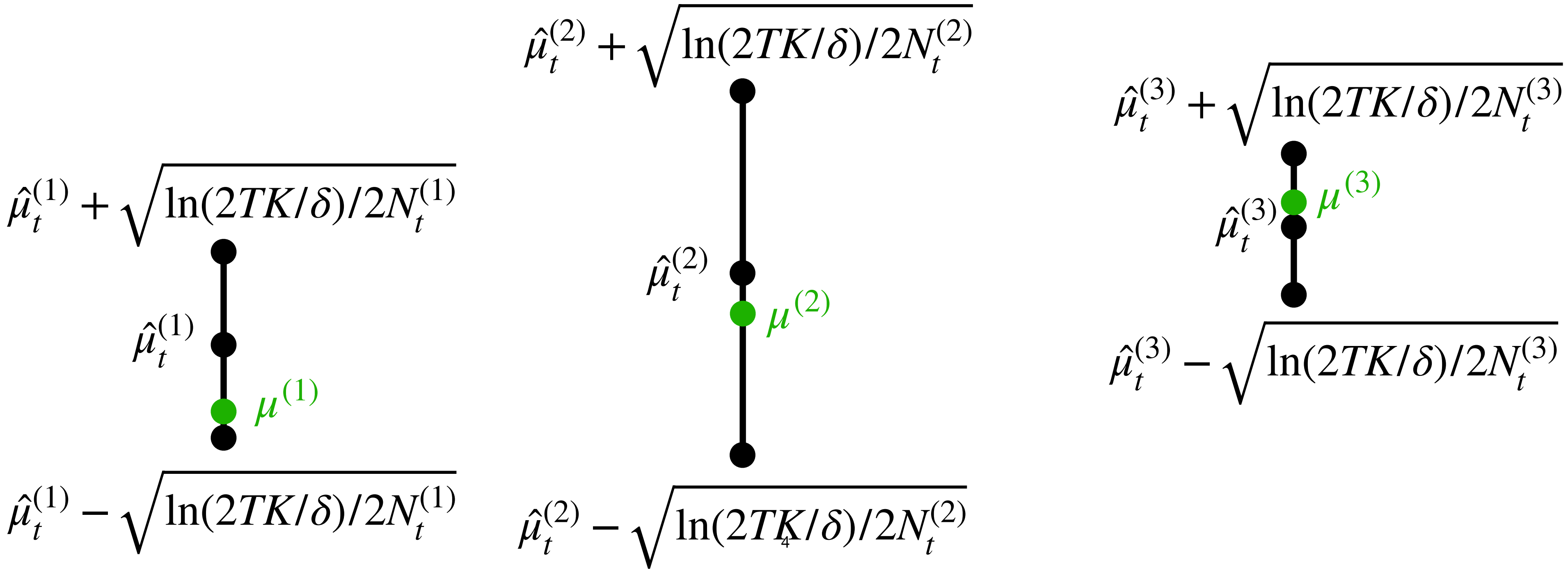

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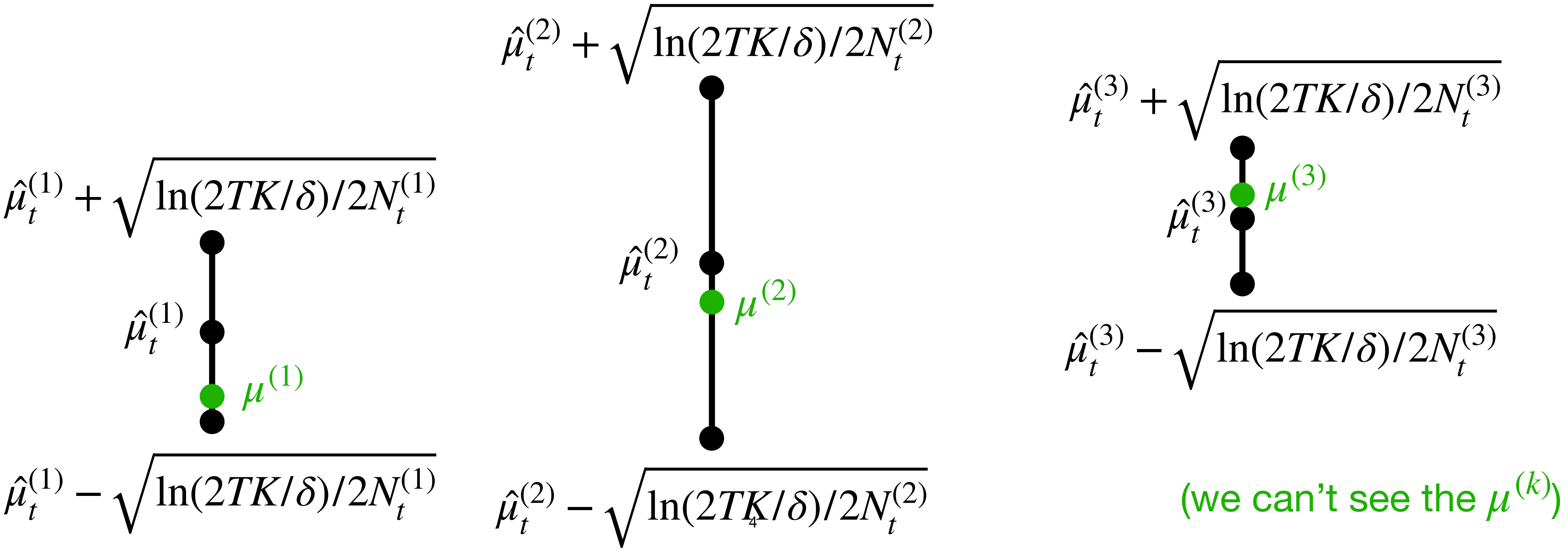

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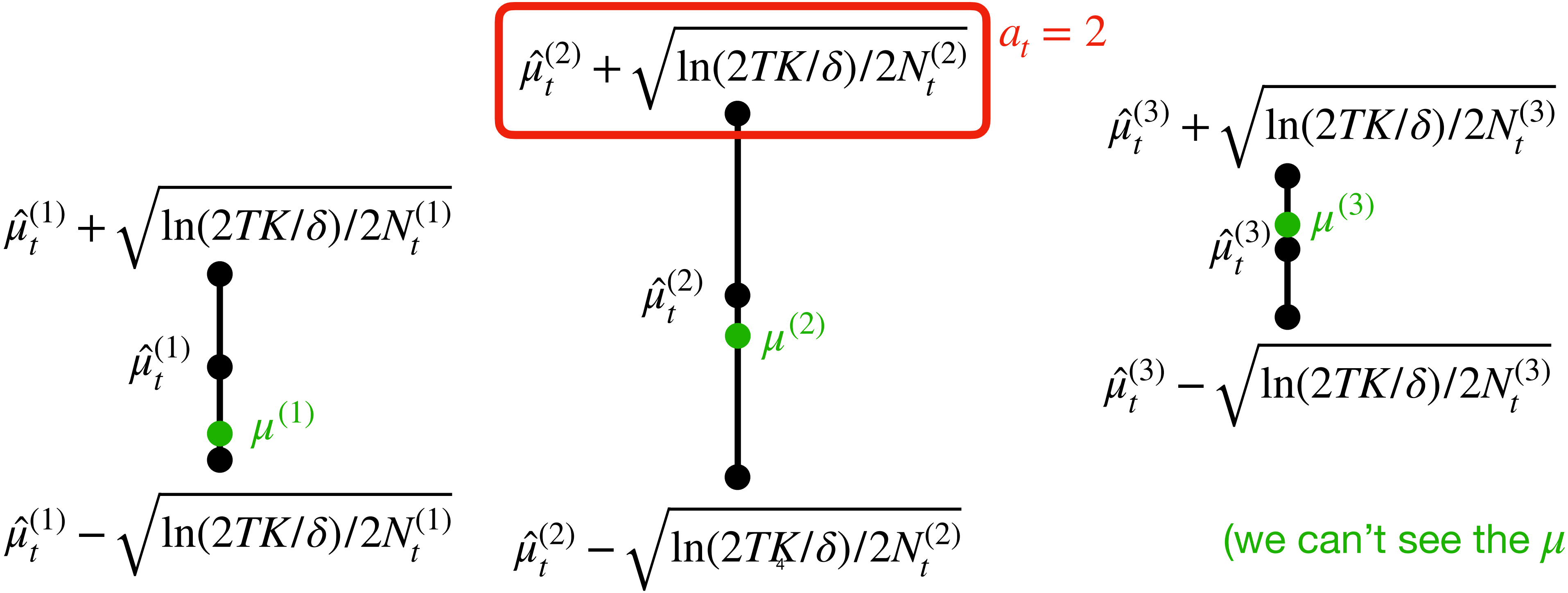

(we can't see the $\mu^{(k)}$)

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Note that the exploration here is *adaptive*, i.e., focused on most promising arms

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Contextual bandit is exactly a MDP with horizon $H = 1$, where x_t is the (singular) state in each episode (so $\mu_0 = \nu_x$)

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Choosing the best model, fitting it, and quantifying uncertainty are essentially problems of supervised learning (for another day)

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$\sum x_\tau r_\tau 1_{\{\}} = \theta \sum x_\tau x_\tau^\top 1_{\{$

proof: $\nabla_\theta \left[\sum_{\tau=0}^{t-1} (r_\tau - x_\tau^\top \theta)^2 1_{\{a_\tau = k\}} \right] = 2 \sum_{\tau=0}^{t-1} x_\tau (r_\tau - x_\tau^\top \theta) 1_{\{a_\tau = k\}} = 0$

Linear model fitting (cont'd)

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$A_t^{(k)}$ must be invertible, which basically requires $N_t^{(k)} \geq d$

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But not using sample mean as estimator, so need something other than Hoeffding

Uncertainty quantification

For UCB, recall that we need confidence bounds on the expected reward of each arm (given context x_t)

Hoeffding was the main tool so far, but it used the fact that our estimate for the expected reward was a sample mean of the rewards we'd seen so far in the same setting (action, context)

With a model, we can use rewards we've seen in other settings → better estimation

But not using sample mean as estimator, so need something other than Hoeffding

Chebyshev's inequality: for a mean-zero random variable Y,

$$|Y| \leq \beta\sqrt{\mathbb{E}[Y^2]} \quad \text{with probability } \geq 1 - \underbrace{1/\beta^2}$$

$|Y| \leq \frac{1}{\sqrt{\delta}}\sqrt{E[Y^2]}$ w.p. $\geq 1-\delta$

$\delta = \frac{1}{\beta^2} \Rightarrow \beta = \frac{1}{\sqrt{\delta}}$

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Let $w_t = r_t - \mathbb{E}_{r \sim \nu^{(k)}(x_t)}[r] = r_t - x_t^\top \theta^{(k)}$, and we derive a useful expression for $\hat{\theta}_t^{(k)}$:

$$\hat{\theta}_t^{(k)} = (A_t^{(k)})^{-1} \sum_{\tau=0}^{t-1} x_\tau r_\tau \mathbb{1}_{\{a_\tau = k\}} = (A_t^{(k)})^{-1} \sum_{\tau=0}^{t-1} x_\tau (x_\tau^\top \theta^{(k)} + w_\tau) \mathbb{1}_{\{a_\tau = k\}}$$

$$= \cancel{(A_t^{(k)})^{-1} \underbrace{\sum_{\tau=0}^{t-1} x_\tau x_\tau^\top \mathbb{1}_{\{a_\tau = k\}}}_{A_t^{(k)}}} \theta^{(k)} + (A_t^{(k)})^{-1} \sum_{\tau=0}^{t-1} x_\tau w_\tau \mathbb{1}_{\{a_\tau = k\}}$$

Uncertainty quantification (cont'd)

Recall: $\hat{\theta}_t^{(k)} = \theta^{(k)} + (A_t^{(k)})^{-1} \sum_{\tau=0}^{t-1} x_\tau 1_{\{a_\tau = k\}} w_\tau$

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Assume for simplicity that we are doing pure exploration, so the actions at each time step are totally independent of everything else.

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Large when $N_t^{(k)}$ small or x_t not aligned with historical data

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Makes $A_t^{(k)}$ invertible always, and it turns out a bound just like Chebyshev's applies (with more details and a much more complicated proof, which we won't get into)

LinUCB algorithm

For $t = 0 \rightarrow T - 1$

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Can prove $\tilde{O}(\sqrt{T})$ regret bound

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Both cases allow a version of linUCB by extension of the same ideas: fit coefficients via least squares and use Chebyshev-like uncertainty quantification to get UCB

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iii. The other formulation, with separate $A_t^{(k)}$ and $\hat{\theta}_t^{(k)}$, is called disjointed

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But in principle, there is no "free lunch", i.e., the hardness of the problem now transfers over to choosing a good model (a bad model will lead to bad performance)

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1-minute feedback form: https://bit.ly/3RHtIxy