Exploration: UCB-VI

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CS/Stat 184: Introduction to Reinforcement Learning
Fall 2022

Today

- Recap
- Why we don’t want to treat MDPs as big contextual bandits
- UCB-VI for tabular MDPs
- UCB-VI for linear MDPs

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Without context:

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For $t = 0, \ldots, T-1$:

Choose the arm with the highest upper confidence bound, i.e.,

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V/Q functions in Finite horizon MDP

$$V_h^\pi(s) = \mathbb{E}\left[\sum_{\tau=h}^{H-1} r(s_\tau, a_\tau) \,\middle|\, s_h = s\right]$$

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Bellman Consistency Equation:

$$Q_h^\pi(s,a) = r(s,a) + \mathbb{E}_{s' \sim P(s,a)}\left[V_{h+1}^\pi(s')\right]$$

Compute Optimal Policy via VI/DP

VI = DP is a backwards in time approach for computing the optimal policy:

$$\pi^\star = \{\pi_0^\star, \pi_1^\star, \ldots, \pi_{H-1}^\star\}$$

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Summary on Finite horizon MDP

$$\mathscr{M} = \{S, A, r, P, H\},$$
$$r : S \times A \mapsto [0,1], \quad H \in \mathbb{N}, \quad P : S \times A \mapsto \Delta(S)$$

Comparing to the infinite horizon, discounted MDP:

1. Policy will be time dependent
2. DP takes H steps to compute $\pi^\star$
   - total computation time is $O(H\,|S|^2\,|A|)$
   - no need to use contraction argument and no discount factor
3. Extension to non-stationary setting works immediately:
   (i.e. with a non-stationary transition model: $P_0(s' \mid s, a), P_1(s' \mid s, a), \ldots P_{H-1}(s' \mid s, a)$)

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This seems bad, so are MDPs just super hard or can we do better?

An example of MDP as contextual bandit

$$S = \{a, b\}, \quad A = \{1,2\}, \quad H = 2$$

All state transitions happen with probability 1/2 for all actions

Reward function:
$$r(a,1) = r(b,1) = 0$$
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$|A|^{|S|H} = 2^4 = 16$

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Optimistic planning with learned model: $\pi^n = \text{Value-Iter}\left(\{ \widehat{P}_h^n, r_h + b_h^n \}_{h=1}^{H-1} \right)$

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Collect a new trajectory by executing π^n in the real world $\{P_h\}_{h=0}^{H-1}$ starting from s_0

Model Estimation

Let us consider the **very beginning** of episode n:

$$\mathscr{D}_h^n = \{s_h^i, a_h^i, s_{h+1}^i\}_{i=1}^{n-1}, \forall h$$

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Estimate model $\widehat{P}_h^n(s' \mid s,a), \forall s,a,s',h$:

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UCBVI: Put All Together

For $n = 1 \to N$:

1. Set $N_h^n(s,a) = \sum_{i=1}^{n-1} \mathbf{1}\{(s_h^i, a_h^i) = (s,a)\}, \forall s, a, h$
2. Set $N_h^n(s,a,s') = \sum_{i=1}^{n-1} \mathbf{1}\{(s_h^i, a_h^i, s_{h+1}^i) = (s,a,s')\}, \forall s, a, a', h$
3. Estimate $\widehat{P}^n$: $\widehat{P}_h^n(s' \mid s,a) = \frac{N_h^n(s,a,s')}{N_h^n(s,a)}, \forall s, a, s', h$
4. Plan: $\pi^n = VI\left(\{\widehat{P}_h^n, r_h + b_h^n\}_h\right)$, with $b_h^n(s,a) = cH\sqrt{\frac{\log(SAHN/\delta)}{N_h^n(s,a)}}$
5. Execute π^n : $\{s_0^n, a_0^n, r_0^n, \ldots, s_{H-1}^n, a_{H-1}^n, r_{H-1}^n, s_H^n\}$

High-level Idea: Exploration Exploitation Tradeoff

Upper bound per-episode regret: $V_0^\star(s_0) - V_0^{\pi^n}(s_0) \leq \widehat{V}_0^n(s_0) - V_0^{\pi^n}(s_0)$

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High-level Idea: Exploration Exploitation Tradeoff

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Today

- ✓ Recap
- ✓ Why we don’t want to treat MDPs as big contextual bandits
- ✓ UCB-VI for tabular MDPs
- UCB-VI for linear MDPs

Linear MDP Definition

Finite horizon time-dependent episodic MDP $\mathscr{M} = \{S, A, H, \{r\}_h, \{P\}_h, s_0\}$

S & A could be large or even continuous, hence poly(S,A) is not acceptable

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Feature map ϕ is known to the learner!

(We assume reward is known, i.e., $\theta^\star$ is known)

Planning in Linear MDP: Value Iteration

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Indeed we can show that $Q_h^\pi(\cdot,\cdot)$

Is linear with respect to ϕ as well, for any π, h

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3. Plan: $\pi^{n+1} = \text{Value-Iter}\left(\{\widehat{P}^n\}_h, \{r_h + b_h^n\}\right)$

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How to choose $b_h^n(s,a)$?

Chebyshev-like approach, similar to in linUCB:

$$b_h^n(s,a) = \beta\sqrt{\phi(s,a)^\top (A_h^n)^{-1}\phi(s,a)}, \quad \beta = \widetilde{O}(dH)$$

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For $n = 1 \to N$:

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No S, A dependence!

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UCB-VI algorithm for tabular MDPs

- Uses UCB idea, but leverages MDP structure

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1-minute feedback form: https://bit.ly/3RHtlxy

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$$\widehat{V}^n_0(s_0) - V^{\pi^n}_0(s_0) = \widehat{Q}^n_0(s_0, \pi^n(s_0)) - Q^{\pi^n}_0(s_0, \pi^n(s_0))$$

$$\le r_0(s_0, \pi^n(s_0)) + b^n_h(s_0, \pi^n(s_0)) + \widehat{P}^n_0(\cdot \mid s_0, \pi^n(s_0)) \cdot \widehat{V}^n_1 - r_0(s_0, \pi^n(s_0)) - P_0(\cdot \mid s_0, \pi^n(s_0)) \cdot V^{\pi^n}_1$$

$$= b^n_h(s_0, \pi^n(s_0)) + \widehat{P}^n_0(\cdot \mid s_0, \pi^n(s_0)) \cdot \widehat{V}^n_1 - P_0(\cdot \mid s_0, \pi^n(s_0)) \cdot V^{\pi^n}_1$$

Simulation Lemma

$$\widehat{V}^n_H(s) = 0, \quad \widehat{Q}^n_h(s,a) = \min\left\{ r_h(s,a) + b^n_h(s,a) + \widehat{P}^n_h(\cdot \mid s,a) \cdot \widehat{V}^n_{h+1}, H \right\}$$

$$\widehat{V}^n_h(s) = \max_a \widehat{Q}^n_h(s,a), \quad \pi^n_h(s) = \arg\max_a \widehat{Q}^n_h(s,a), \forall s$$

Lemma [Simulation lemma]:

$$\widehat{V}^n_0(s_0) - V^{\pi^n}_0(s_0) \le \sum_{h=0}^{H-1} \mathbb{E}_{s,a\sim d^{\pi^n}_h} \left[b^n_h(s,a) + (\widehat{P}^n_h(\cdot \mid s,a) - P_h(\cdot \mid s,a)) \cdot \widehat{V}^n_{h+1} \right]$$

$$\widehat{V}^n_0(s_0) - V^{\pi^n}_0(s_0) = \widehat{Q}^n_0(s_0, \pi^n(s_0)) - Q^{\pi^n}_0(s_0, \pi^n(s_0))$$

$$\le r_0(s_0, \pi^n(s_0)) + b^n_h(s_0, \pi^n(s_0)) + \widehat{P}^n_0(\cdot \mid s_0, \pi^n(s_0)) \cdot \widehat{V}^n_1 - r_0(s_0, \pi^n(s_0)) - P_0(\cdot \mid s_0, \pi^n(s_0)) \cdot V^{\pi^n}_1$$

$$= b^n_h(s_0, \pi^n(s_0)) + \widehat{P}^n_0(\cdot \mid s_0, \pi^n(s_0)) \cdot \widehat{V}^n_1 - P_0(\cdot \mid s_0, \pi^n(s_0)) \cdot V^{\pi^n}_1$$

$$= b^n_h(s_0, \pi^n(s_0)) + \left(\widehat{P}^n_0(\cdot \mid s_0, \pi^n(s_0)) - P_0(\cdot \mid s_0, \pi^n(s_0)) \right) \cdot \widehat{V}^n_1 + P_0(\cdot \mid s_0, \pi^n(s_0)) \cdot \left(\widehat{V}^n_1 - V^{\pi^n}_1 \right)$$

Simulation Lemma

$$\widehat{V}^n_H(s) = 0, \quad \widehat{Q}^n_h(s,a) = \min\left\{ r_h(s,a) + b^n_h(s,a) + \widehat{P}^n_h(\cdot \mid s,a) \cdot \widehat{V}^n_{h+1}, H \right\}$$

$$\widehat{V}^n_h(s) = \max_a \widehat{Q}^n_h(s,a), \quad \pi^n_h(s) = \arg\max_a \widehat{Q}^n_h(s,a), \forall s$$

Lemma [Simulation lemma]:

$$\widehat{V}^n_0(s_0) - V^{\pi^n}_0(s_0) \le \sum_{h=0}^{H-1} \mathbb{E}_{s,a \sim d^{\pi^n}_h} \left[b^n_h(s,a) + (\widehat{P}^n_h(\cdot \mid s,a) - P_h(\cdot \mid s,a)) \cdot \widehat{V}^n_{h+1} \right]$$

$$\widehat{V}^n_0(s_0) - V^{\pi^n}_0(s_0) = \widehat{Q}^n_0(s_0, \pi^n(s_0)) - Q^{\pi^n}_0(s_0, \pi^n(s_0))$$

$$\le r_0(s_0, \pi^n(s_0)) + b^n_h(s_0, \pi^n(s_0)) + \widehat{P}^n_0(\cdot \mid s_0, \pi^n(s_0)) \cdot \widehat{V}^n_1 - r_0(s_0, \pi^n(s_0)) - P_0(\cdot \mid s_0, \pi^n(s_0)) \cdot V^{\pi^n}_1$$

$$= b^n_h(s_0, \pi^n(s_0)) + \widehat{P}^n_0(\cdot \mid s_0, \pi^n(s_0)) \cdot \widehat{V}^n_1 - P_0(\cdot \mid s_0, \pi^n(s_0)) \cdot V^{\pi^n}_1$$

$$= b^n_h(s_0, \pi^n(s_0)) + \left(\widehat{P}^n_0(\cdot \mid s_0, \pi^n(s_0)) - P_0(\cdot \mid s_0, \pi^n(s_0)) \right) \cdot \widehat{V}^n_1 + P_0(\cdot \mid s_0, \pi^n(s_0)) \cdot \left(\widehat{V}^n_1 - V^{\pi^n}_1 \right)$$

$$= \sum_{h=0}^{H-1} \mathbb{E}_{s,a \sim d^{\pi^n}_h} \left[b^n_h(s,a) + (\widehat{P}^n_h(\cdot \mid s,a) - P_h(\cdot \mid s,a)) \cdot \widehat{V}^n_{h+1} \right]$$