Bandits: Explore-Then-Commit and ε -greedy

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CS/Stat 184: Introduction to Reinforcement Learning Fall 2022

Today

- Recap
- Explore-then-commit (ETC)
- *ɛ*-greedy

• Reinforcement learning is an *interactive* form of machine learning

- Applicable whenever you want to learn to do something better
- One component is learning while acting: exploration vs exploitation
- Other component is optimization

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 - Exemplify first component (exploration vs exploitation)
 - Pure greedy not much better than pure exploration (linear regret)
- Today: let's do better than linear regret!

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$$T\mu^{\star} - \sum_{t=0}^{T-1} \mu_{a_t} = \sum_{t=0}^{T-1} (\mu^{\star} - \mu_{a_t})$$

Expected regret at time t
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3. Why is linear regret bad? \Rightarrow average regret := $\frac{\text{Regret}_T}{T} \neq 0$

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Plan: (1) try each arm <u>multiple</u> times, (2) compute the empirical mean of each arm, (3) commit to the one that has the highest empirical mean

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Q: how to set N_{e} ?

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- 4. Minimize our upper-bound over N_e

Hoeffding inequality

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Then with probability at least $1 - \delta$,

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- Why is this true? Full proof beyond course scope, but intuition easier...

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- Numerator is because Gaussian has double-exponential tails, i.e., probability of a deviation from the mean by *x* scales roughly like e^{-x^2} , which, when inverted (i.e., set $\delta = e^{-x^2}$ and solve for *x*) gives $x = \sqrt{\ln(1/\delta)}$

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- Don't worry too much about the extra 2's... CLT is only approximate!

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regret at each step of exploitation phase = $\mu_{k^{\star}} - \mu_{\hat{k}}$

$$M_{k} + -\mu_{k} = M_{k} - \mu_{k} + (\hat{\mu}_{k} - \hat{\mu}_{k}) + (\hat{\mu}_{k} - \hat{\mu}_{k}) + (\hat{\mu}_{k} - \hat{\mu}_{k}) + (\hat{\mu}_{k} - \mu_{k}) + (\hat{\mu}_{$$

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Minimize over N_e : (won't bore you with algebra)

optimal
$$N_{e} = \left(\frac{T\sqrt{\ln(2K/\delta)/2}}{K}\right)^{2/3}$$

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(A bit more algebra to plug optimal N_e into Regret_T equation above)

$$\Rightarrow \operatorname{\mathsf{Regret}}_{T} \leq 3T^{2/3} (K \ln(2K/\delta)/2)^{1/3} \qquad \checkmark /\rho \geq (-\delta)^{1/3}$$

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Sample $E_t \sim \text{Bernoulli}(\varepsilon)$

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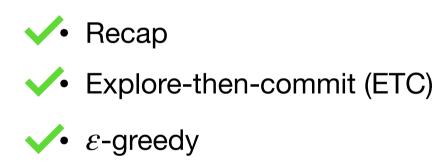
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- Regret rate (ignoring log factors) is the same as ETC, but holds for <u>all</u> *t*, not just the full time horizon *T*
- Nothing in ε -greedy (including ε_t above) depends on T, so don't need to know horizon!

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- balance exploration with exploitation
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1-minute feedback form: https://forms.gle/2mKHGRMCpFTRMQqd8

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Worked for ETC b/c exploration phase was i.i.d., but in general the rewards from a given arm are *not* i.i.d. due to adaptivity of action selections

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Notation:

$\begin{array}{l} \text{Notation:}\\ \text{Let } N_t^{(k)} = \sum_{\tau=0}^{t-1} \mathbf{1}_{\{a_{\tau}=k\}} \text{ be the number of times arm } k \text{ is pulled before time } t \end{array}$

Constructing confidence intervals Notation: Let $N_t^{(k)} = \sum_{k=1}^{t-1} 1_{\{a_r=k\}}$ be the number of times arm k is pulled before time t $\tau = 0$ Let $\hat{\mu}_t^{(k)} = \frac{1}{N_t^{(k)}} \sum_{\tau=0}^{t-1} 1_{\{a_\tau = k\}} r_\tau$ be the sample mean reward of arm k up to time t

Constructing confidence intervals

Notation:

Let $N_t^{(k)} = \sum_{\tau=0}^{t-1} 1_{\{a_\tau=k\}}$ be the number of times arm k is pulled before time tLet $\hat{\mu}_t^{(k)} = \frac{1}{N_t^{(k)}} \sum_{\tau=0}^{t-1} 1_{\{a_\tau=k\}} r_{\tau}$ be the sample mean reward of arm k up to time t

So want Hoeffding to give us something like

$$\left|\hat{\mu}_{t}^{(k)} - \mu\right| \leq \sqrt{\frac{\ln(2/\delta)}{2N_{t}^{(k)}}} \text{ w/p } 1 - \delta$$

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But this is generally FALSE

(unless a_t chosen very simply, like exploration phase of ETC)

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Then we can think of every time we pull arm k, just pulling the next $\tilde{r}_i^{(k)}$ off this list, i.e., $r_{\tau}^{(k)} \mid a_{\tau} = k$ to simply equal to $\tilde{r}_{N_{\tau}^k}^{(k)}$, and hence $\hat{\mu}_t^{(k)} = \frac{1}{N_t^{(k)}} \sum_{i=0}^{N_t^{(k)}-1} \tilde{r}_i^{(k)}$

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Hoeffding + union bound over
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 $\Rightarrow \mathbb{P}\left(\forall n \le t, |\tilde{\mu}_n^{(k)} - \mu^{(k)}| \le \sqrt{\ln(2t/\delta)/2n} \right) \ge 1 - \delta$

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Summary: to deal with problem of non-i.i.d. rewards that enter into $\hat{\mu}_t^{(k)}$, we used rewards' *conditional* i.i.d. property along with a union bound to get Hoeffding bound that is wider by just a factor of *t* in the log term