Bandits: Upper Confidence Bound Algorithm

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CS/Stat 184: Introduction to Reinforcement Learning

Fall 2022

Today

- Feedback from last lecture
- Recap
- Confidence intervals for the arms
- Upper Confidence Bound (UCB) algorithm
- UCB regret analysis

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4. Use bit.ly and QR code for feedback form: done!

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i.e., $\mathbb{P}(X_i \leq B(\delta)) \geq 1-\delta, \ \forall i \quad \longrightarrow \quad \mathbb{P}(\forall i, X_i \leq B_n(\delta)) \geq 1-\delta$

$$\mathbb{P}(\forall i, X_i \leq B(\delta)) = 1 - \mathbb{P}(\exists i, X_i > B(\delta))$$

$$\text{U.B.} \rightarrow \geq 1 - \sum_{i=1}^{n} \mathbb{P}(X_i > B(\delta)) = 1 - \sum_{i=1}^{n} \underbrace{(1 - \overbrace{\mathbb{P}(X_i \leq B(\delta))}^{\geq 1-\delta})}_{\leq \delta} \geq 1 - n\delta$$

$$\delta \rightarrow \frac{\delta}{n} \Rightarrow \mathbb{P}(\forall i, X_i \leq B(\delta/n)) \geq 1 - n\frac{\delta}{n} = 1 - \delta$$

$$\Rightarrow B_n(\delta) = B(\delta/n) \ \checkmark$$

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Recall Hoeffding inequality:

Sample mean of N i.i.d. samples on $[0,1]$ satisfies

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Worked for ETC b/c exploration phase was i.i.d., but in general the rewards from a given arm are *not* i.i.d. due to adaptivity of action selections

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But this is generally FALSE

(unless a_t chosen very simply, like exploration phase of ETC)

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i.e., $r_\tau \mid a_\tau = k$ to simply equal to $\tilde{r}_{N_\tau^k}^{(k)}$, and hence $\hat{\mu}_t^{(k)} = \frac{1}{N_t^{(k)}} \sum_{i=0}^{N_t^{(k)}-1} \tilde{r}_i^{(k)}$

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<u>Summary</u>: to deal with problem of non-i.i.d. rewards that enter into $\hat{\mu}_t^{(k)}$, we used rewards' *conditional* i.i.d. property along with a union bound to get Hoeffding bound that is wider by just a factor of t in the log term

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So we have a valid $(1-\delta)$ confidence interval (CI) for $\mu^{(k)}$ at time t from last equation:

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Valid for any bandit algorithm!
Of independent statistical interest
for interpreting results

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By same argument made in ETC analysis, union bound over K makes coverage uniform over k:

$$\mathbb{P}\left(\forall k \leq K, t < T,\ |\hat{\mu}_t^{(k)} - \mu^{(k)}| \leq \sqrt{\ln(2TK/\delta)/2N_t^{(k)}}\right) \geq 1-\delta$$

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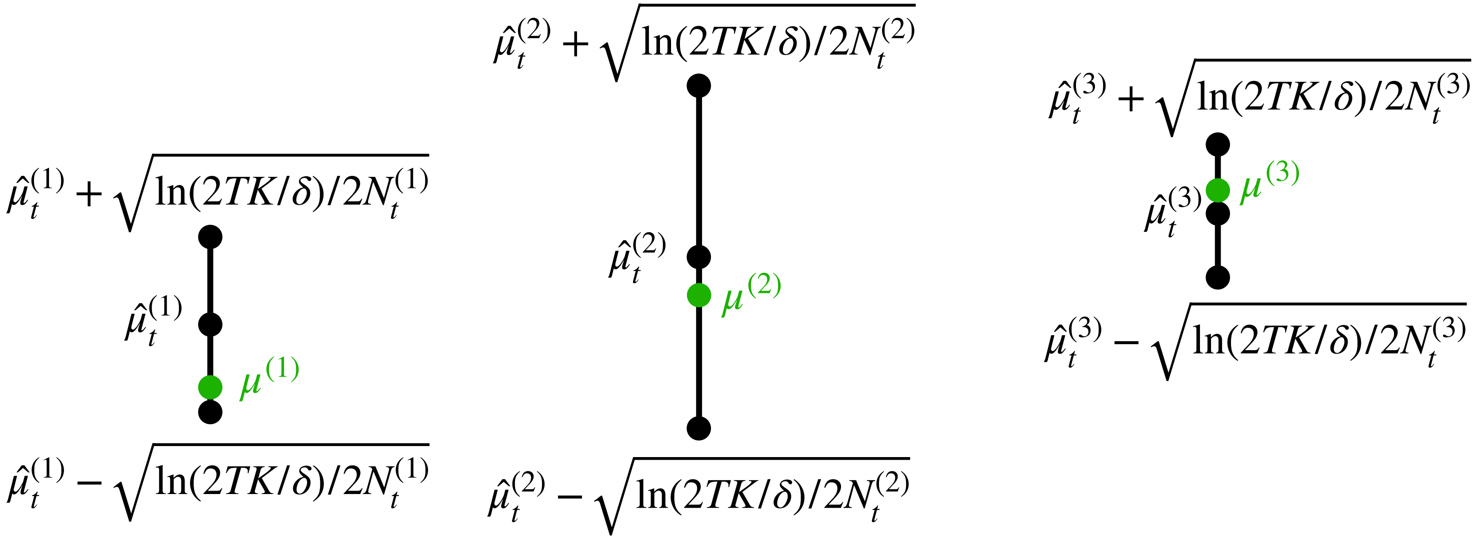

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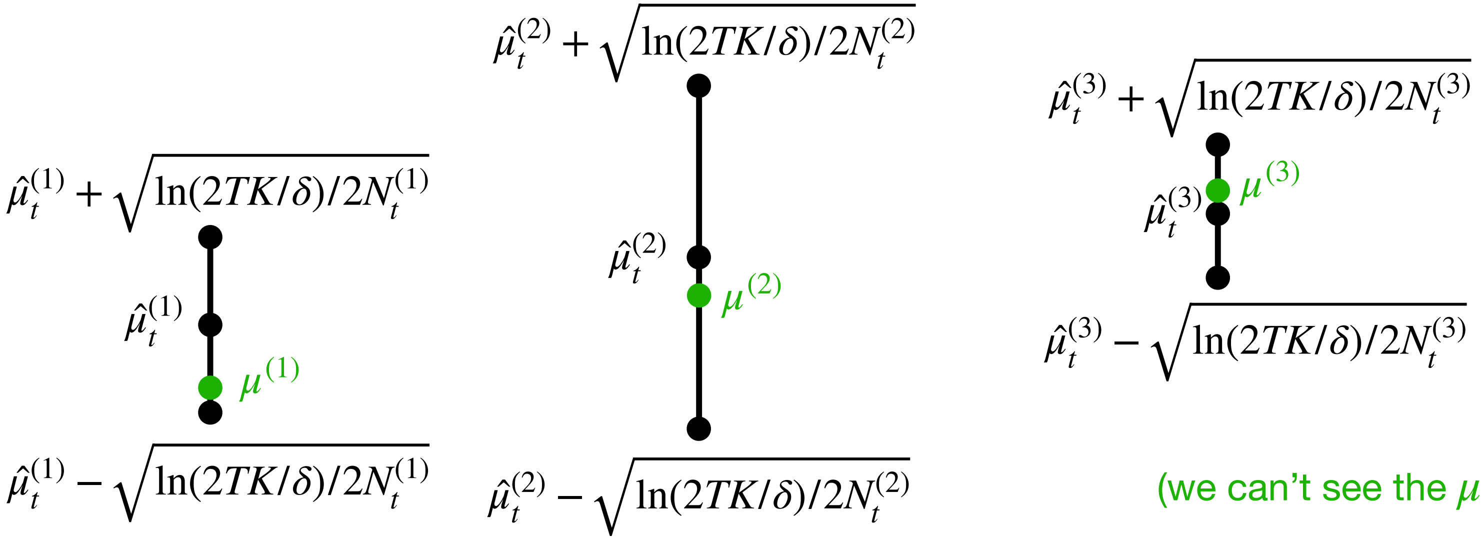

(we can't see the $\mu^{(k)}$)

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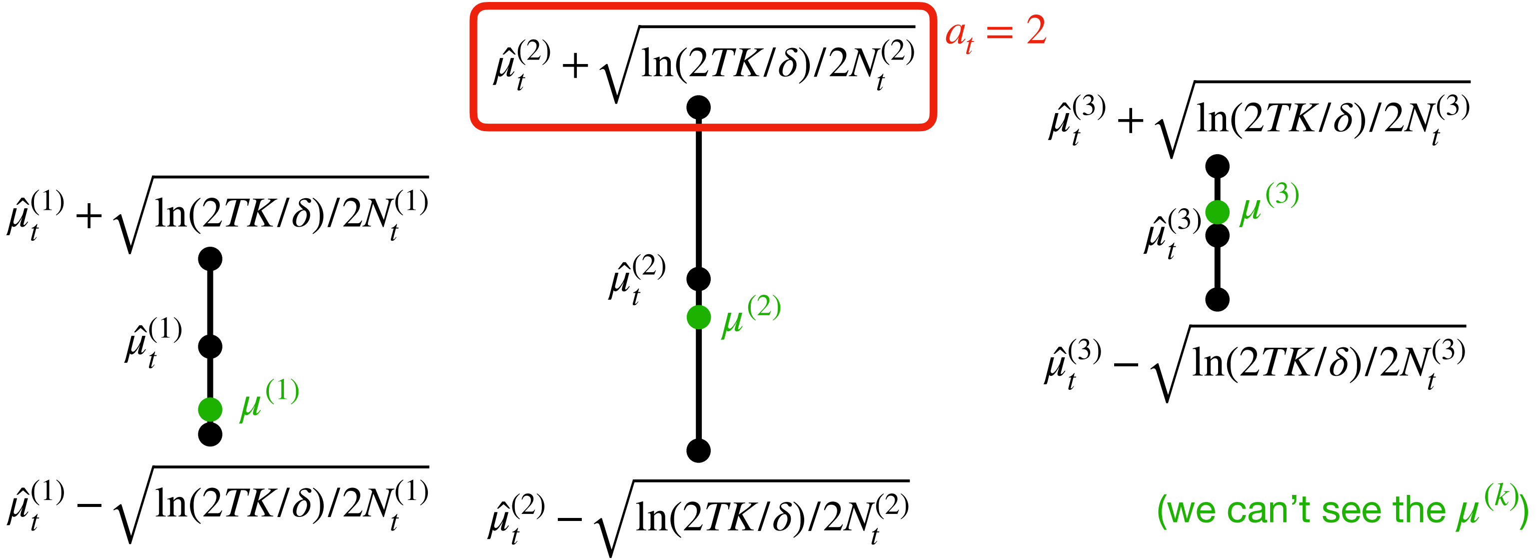

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Note that the exploration here is *adaptive*, i.e., focused on most promising arms

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$$B_t^{(k)} = \sqrt{\ln(2KT/\delta)/2N_t^{(k)}}$$

UCB regret at each time step

Recall $k^\star$ is optimal arm, so $\mu^{(k^\star)}$ is true best arm mean. Thus time step t regret is:

$$\mu^{(k^\star)} - \mu^{(a_t)} \leq \hat{\mu}_t^{(k^*)} + B_t^{(k^*)} - \mu^{(a_t)}$$

$$\leq \hat{\mu}_t^{(a_t)} + B_t^{(a_t)} - \mu^{(a_t)}$$

$$\leq B_t^{(a_t)} + B_t^{(a_t)} = 2 B_t^{(a_t)}$$

all w/p $\geq 1-\delta$, $\boxed{\text{uniformly over } t \leq T}$

Sum of UCB per-time-step regrets

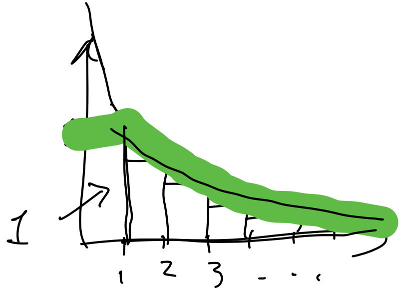
1. per-time-step regret bound $\mu^{(k^\star)} - \mu^{(a_t)} \le \sqrt{2\ln(2KT/\delta)/N_t^{(a_t)}} = 2B_t^{(a_t)}$ w/p $1-\delta$

2. $\text{Regret}_T \le \sum_{t=0}^{T-1}\sqrt{2\ln(2KT/\delta)/N_t^{(a_t)}} = \sqrt{2\ln(2KT/\delta)}\sum_{t=0}^{T-1}\sqrt{\frac{1}{N_t^{(a_t)}}}$

$$\sum_{t=0}^{T-1}\frac{1}{\sqrt{N_t^{(a_t)}}} = \sum_{t=0}^{T-1}\sum_{k=0}^{K}1_{\{a_t=k\}}\sqrt{\frac{1}{N_t^{(k)}}} = \sum_{k=1}^{K}\sum_{n=1}^{N_T^{(k)}}\sqrt{\frac{1}{n}} \le K\sum_{n=1}^{T}\frac{1}{\sqrt{n}}$$

$$\sum_{n=1}^{T}\frac{1}{\sqrt{n}} \le 1 + \int_1^T \frac{1}{\sqrt{x}}\,dx = 1 + 2\sqrt{x}\Big|_{x=1}^{x=T}$$

$$= 2\sqrt{T} - 1 \le 2\sqrt{T}$$



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Finally, putting it all together, we get:

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In fact, a more sophisticated analysis can get: $\text{Regret}_T = \tilde{O}(\sqrt{KT}) \quad \text{w/p } 1-\delta$

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1-minute feedback form: https://bit.ly/3RHtIxy