Bandits: Regret Lower Bound and Instance-Dependent Regret

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CS/Stat 184: Introduction to Reinforcement Learning Fall 2022

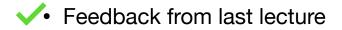
Today

- Feedback from last lecture
- Recap
- Regret *lower* bound
- Instance-dependent regret

Feedback from feedback forms

- 1. Thank you to everyone who filled out the forms!
- 2. Main feedback: pace was good!
- 3. Pre-lecture posted lecture notes shouldn't maintain breaks within slides

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Recap

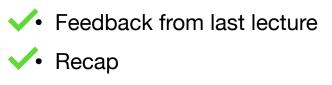
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- Can we do even better?

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An oracle has access to extra information not available to bandit algorithms.

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If we can show that oracle can't do better than some rate, then no algorithm can

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- 4. Thus, we should expect to sample \tilde{k} roughly as often as k^{\star} , which is at best roughly T/2 times (if we ignore any other arms)
- 5. Finally, since the regret incurred each time we pull arm \tilde{k} is $1/\sqrt{T}$, and we pull it T/2 times, we get a regret lower bound of $1/\sqrt{T} \times T/2 = \Omega(\sqrt{T})$

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Additionally: oracle chooses all a_t after seeing all arm rewards up to time T (one decision point makes theory easier)

Oracle strategy

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Oracle strategy (cont'd)

Best strategy in terms of maximizing
$$\sum_{t=0}^{T-1} \mu^{(a_t)}$$
 (i.e., minimizing Regret_T), is to choose every $a_t = \hat{k}_T = \arg \max_{k \in 1, ..., K} \hat{\mu}_T^{(k)}$, since \hat{k}_T is the oracle's best guess of k^*

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This was not mathematically rigorous, but hopefully you can see why this strategy is the best strategy the oracle could employ given the information it has

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Let $C_k := \operatorname{Var}_{r \sim \nu^{(k^*)}}(r) + \operatorname{Var}_{r \sim \nu^{(k)}}(r)$ and suppose that $\mu^{(k^*)} - \mu^{(k)} = \sqrt{C_k/T}$, then:

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$$\sqrt{\frac{T}{C_k}} (\hat{\mu}_T^{(k^\star)} - \hat{\mu}_T^{(k)}) \approx \mathcal{N}(1,1)$$

$\begin{array}{l} \text{Oracle regret (cont'd)} \\ \text{From previous slide:} \sqrt{\frac{T}{C_k}} (\hat{\mu}_T^{(k^\star)} - \hat{\mu}_T^{(k)}) \approx \mathcal{N}(1,1) \end{array}$

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So if $\mu^{(k^{\star})} - \mu^{(k)} = \sqrt{C_k/T}$ for all $k \neq k^{\star}$, and if all $C_k = C$ for $k \neq k^{\star}$, then

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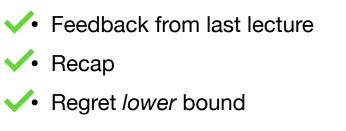
So if $\mu^{(k^*)} - \mu^{(k)} = \sqrt{C_k/T}$ for all $k \neq k^*$, and if all $C_k = C$ for $k \neq k^*$, then $\begin{array}{l} \mathbb{P}(\hat{k}_T \neq k^*) \gtrsim 16 \% \\ \Rightarrow \mathbb{P}(\mu^{(k^*)} - \mu^{(\hat{k}_T)}) = \sqrt{C/T}) \gtrsim 16 \% \end{array}$ Regret_T = $T(\mu^{(k^*)} - \mu^{(\hat{k}_T)})$

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 $\Rightarrow \mathbb{P}(\mu^{(k^{\star})} - \mu^{(\hat{k}_{T})} = \sqrt{C/T}) \gtrsim 16\%$ Regret_T = $T(\mu^{(k^{\star})} - \mu^{(\hat{k}_{T})}) \Rightarrow \mathbb{P}(\text{Regret}_{T} = \sqrt{CT}) \gtrsim 16\%$

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Today



Instance-dependent regret

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- 2. To do that, we'll use the uniform Hoeffding bound to see how often the UCB for k^* is guaranteed (with high probability) to be higher than the UCB for k
- 3. Then we'll multiply $N_T^{(k)}$ by the suboptimality of arm k, and sum this over the arms k to get the total regret

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1-minute feedback form: https://bit.ly/3RHtlxy

