Ascent

Policy Gradient ~~Descent~~

Lucas Janson and Sham Kakade

CS/Stat 184: Introduction to Reinforcement Learning

Fall 2022

Today

- Recap
- Today:
  - How do we learn/compute a good policy in an intractably large MDP?
    - Policy gradient descent is one of the most effective methods.
    - iLQR was our first local search method.

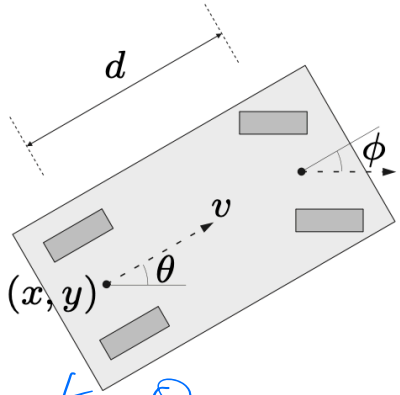
Recap

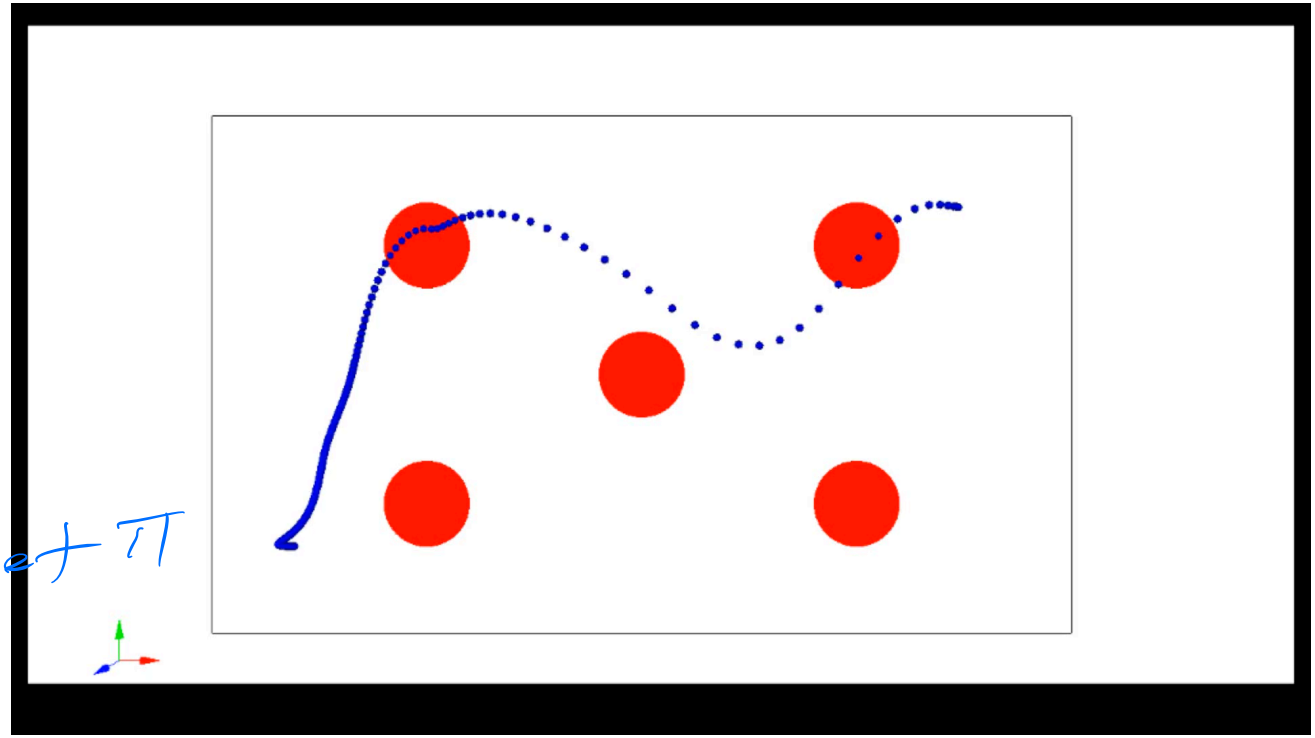
Example:

2-d car navigation

Cost function is designed such that it gets to the goal without colliding with obstacles (in red)

practice: Model Predictive Control (MPC)





for $t = 0, \ldots$

- run iLQR to get π
- $a_t = \pi(s_t)$
- repeat

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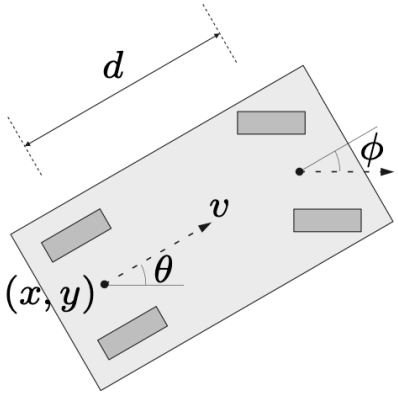
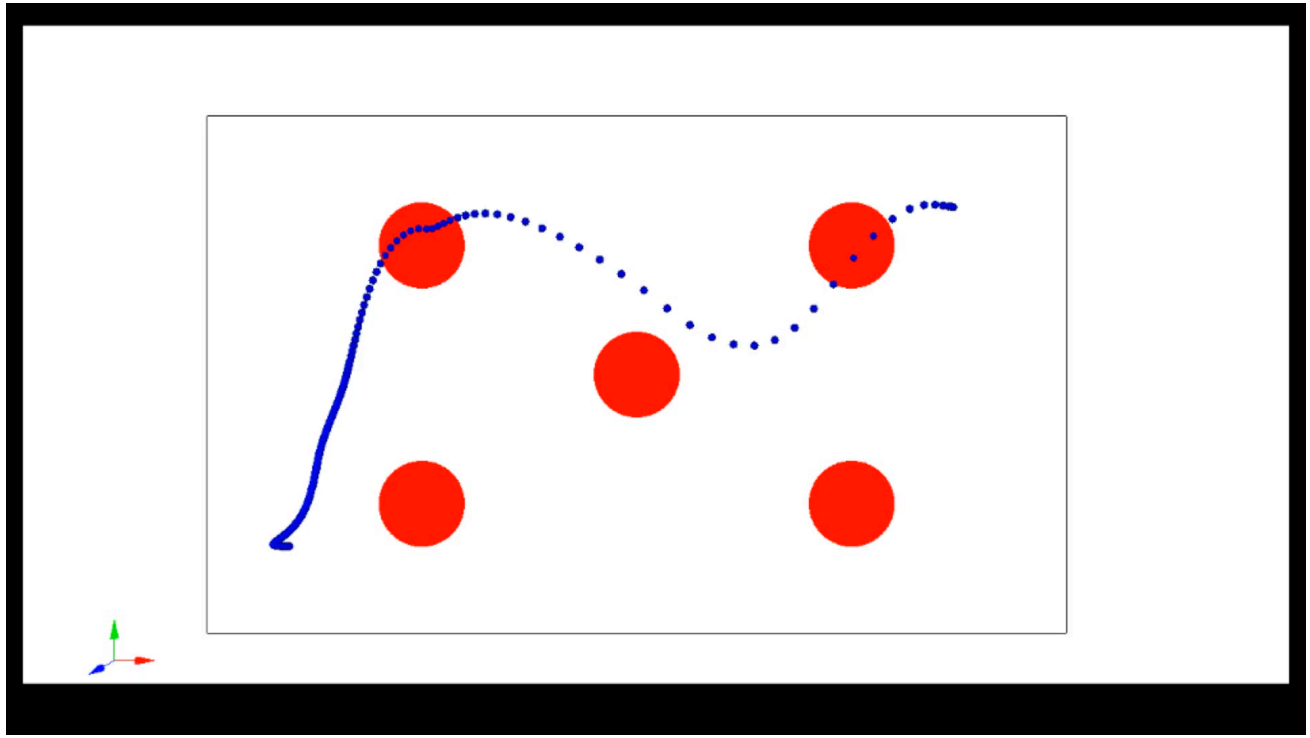


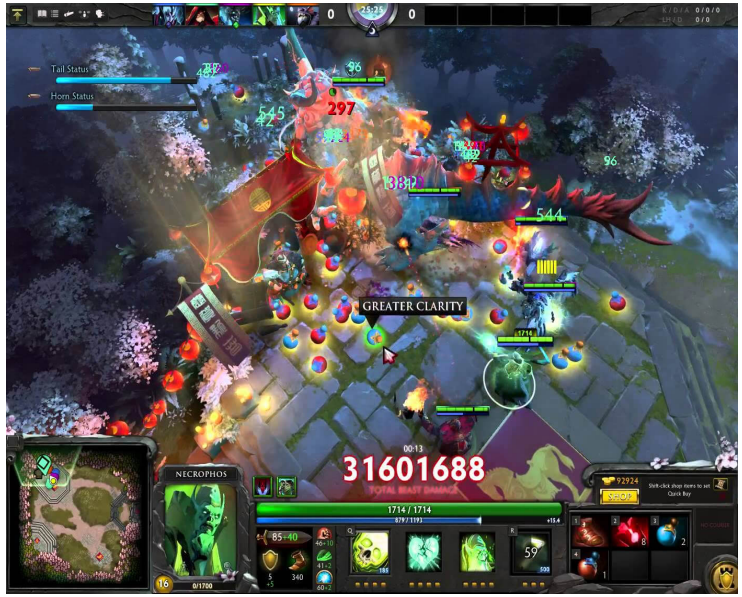


Today:

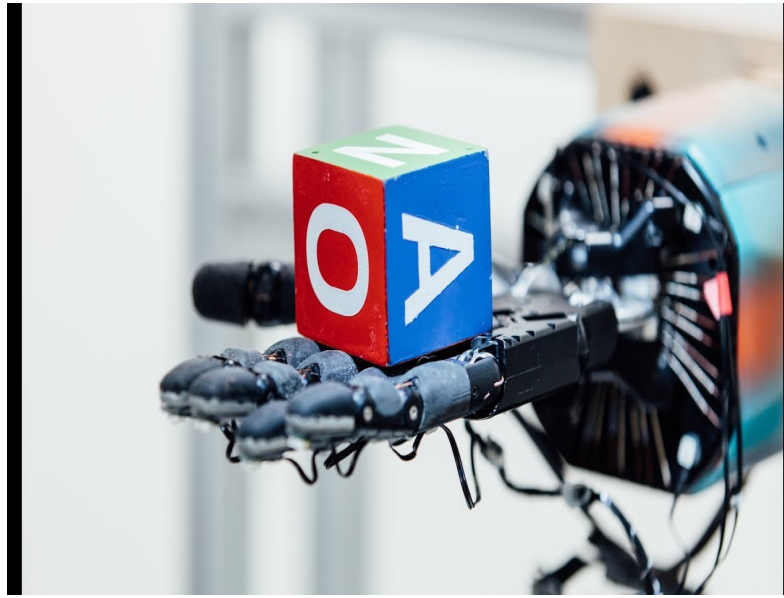
Policy Gradient Descent

Policy Optimization

[AlphaZero, Silver et.al, 17]



[OpenAI Five, 18]



[OpenAI,19]

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Main question for today's lecture:
how to compute the gradient?

Outline for today

1. Recap on Gradient Descent (GD) and Stochastic Gradient Descent (SGD)

2. Warm up: computing gradient using importance weighting

3. Policy Gradient formulations

Gradient Descent

Given an objective function $J(\theta) : \mathbb{R}^d \mapsto \mathbb{R}$, (e.g., $J(\theta) = \mathbb{E}_{x,y}(f_\theta(x) - y)^2$)

GD minimizes the above objective function as follows:

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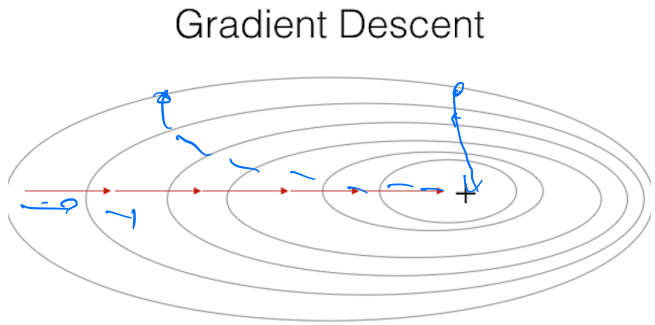

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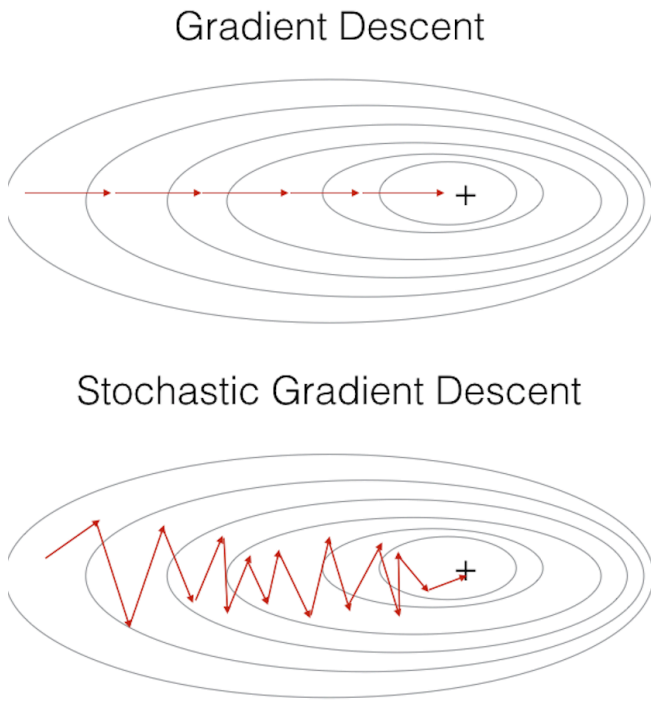

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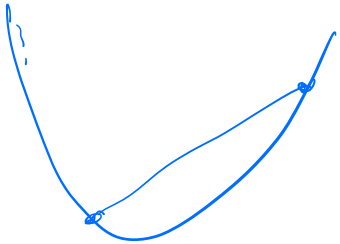
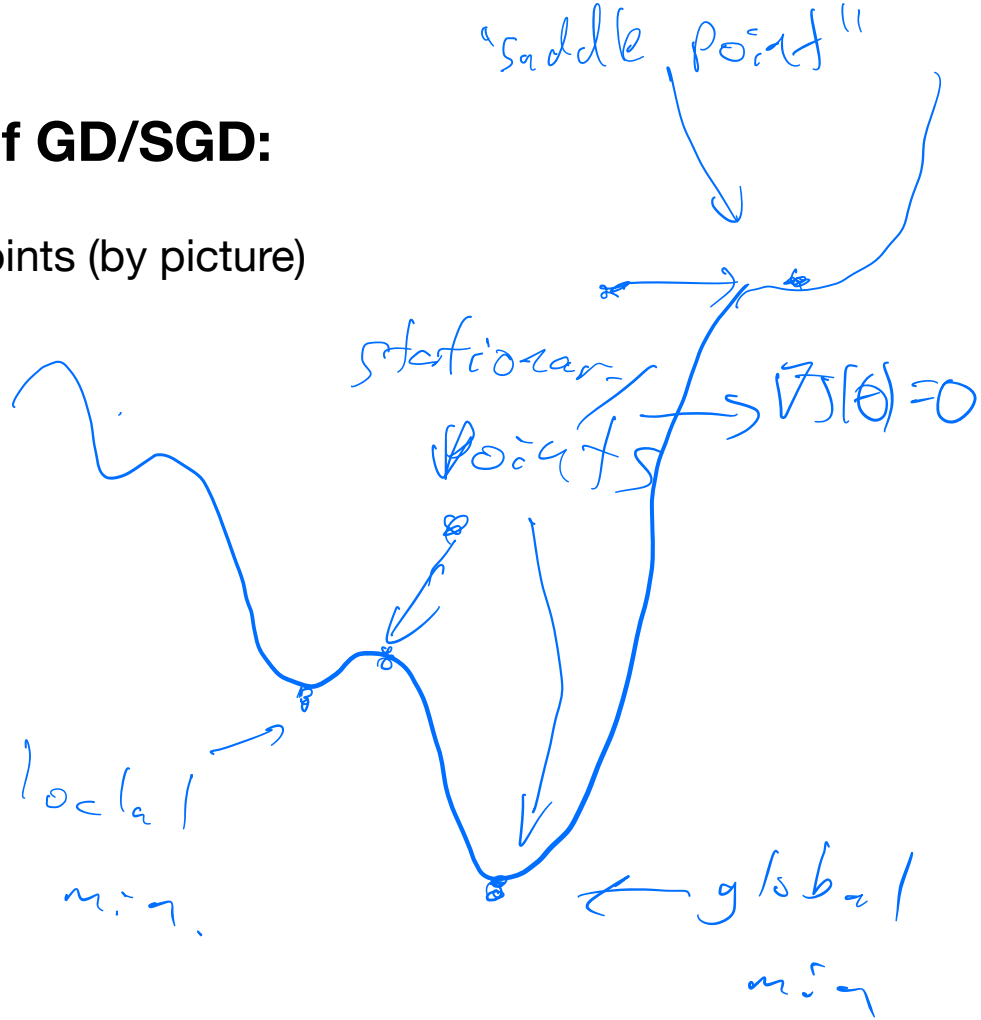


convex function



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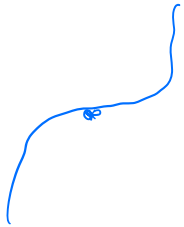

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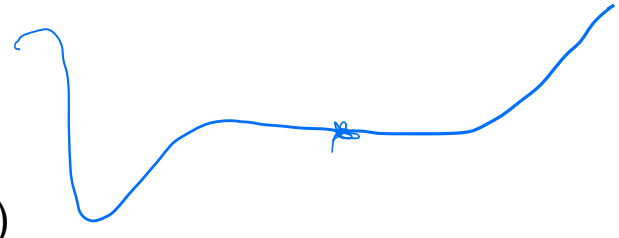

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- For non-convex functions, we hope to find a local minima.

- What we can prove (under mild regularity conditions) is a little weaker:
  - GD (with an appropriate constant learning rate) converges to a saddle point.
  - SGD (with an appropriately decaying learning rate) converges to a saddle point.

$$\hat{J}(\theta) = \frac{1}{N} \sum_{i=1}^{N} \left(f_\theta(x_i) - y_i\right)^2$$

some sort of "nice" stationary point.

SGD: Convergence to a Stationary Point for Nonconvex Functions

- Def of β-smooth: $\|\nabla_\theta J(\theta) - \nabla_\theta J(\theta_0)\|_2 \leq \beta \|\theta - \theta_0\|_2$

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then, in T steps, SGD will find a θ such that: $\|\nabla_\theta J(\theta)\|^2 \leq O\left(\sqrt{M\beta\sigma^2/T}\right).$

$\|\nabla J_\theta\| \leq \left(\frac{\quad}{T}\right)^{\frac{1}{4}}$

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Importance Sampling (and the Likelihood Ratio Method)

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$x_i \sim \rho$

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$$\frac{\nabla_\theta P_\theta(x)}{P_\theta(x)} = \nabla \log P_\theta(x)$$

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$$= \frac{1}{N}\sum_{i=1}^{N} \nabla \log P_\theta(x_i) f(x_i)$$

By setting the sampling distribution $\rho = P_{\theta_0}$

$$\nabla_\theta J(\theta_0) = \mathbb{E}_{x\sim P_{\theta_0}}\left[\nabla_\theta \ln P_{\theta_0}(x) \cdot f(x)\right]$$

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Example and Intuition

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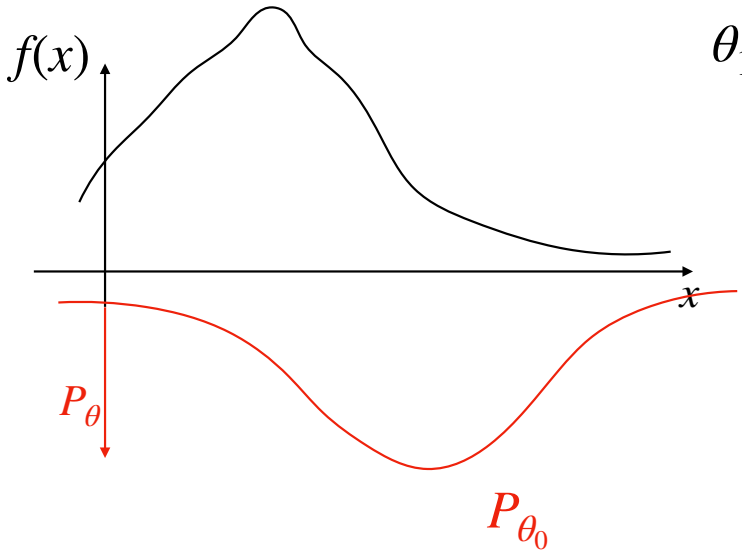

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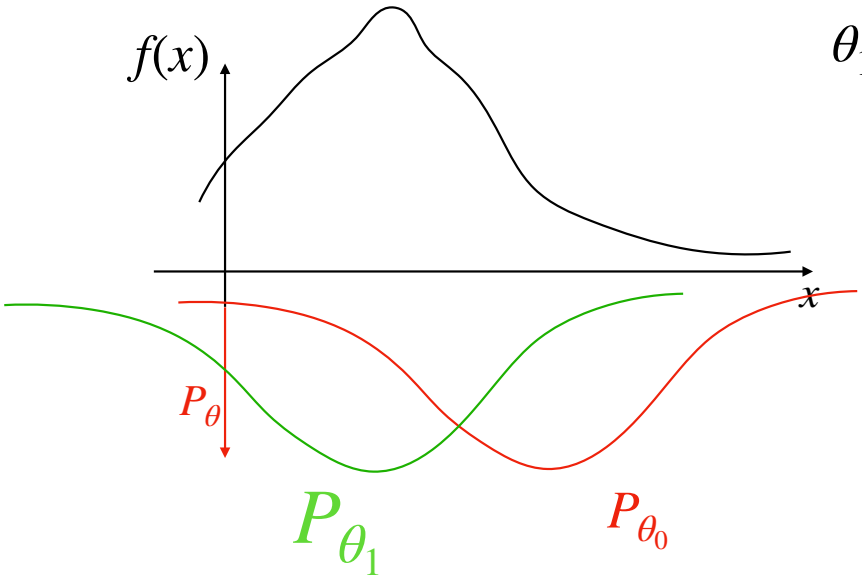

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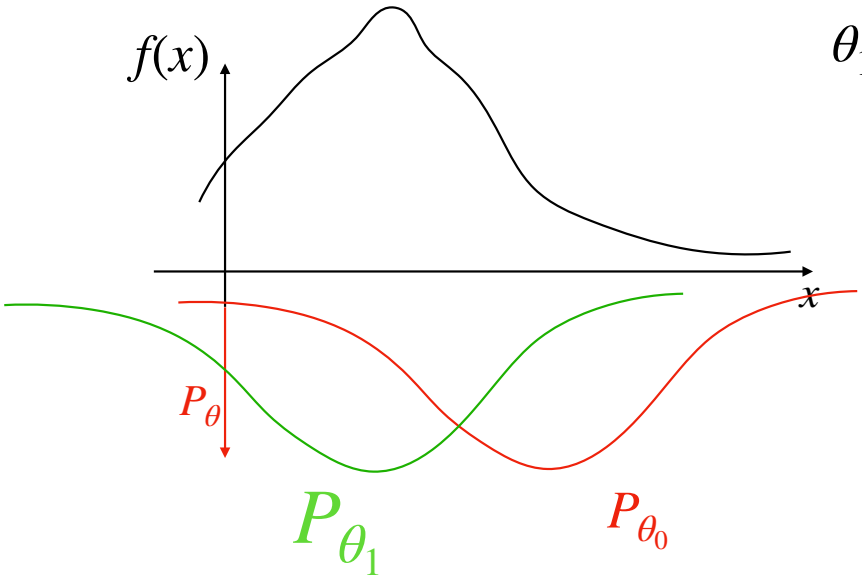
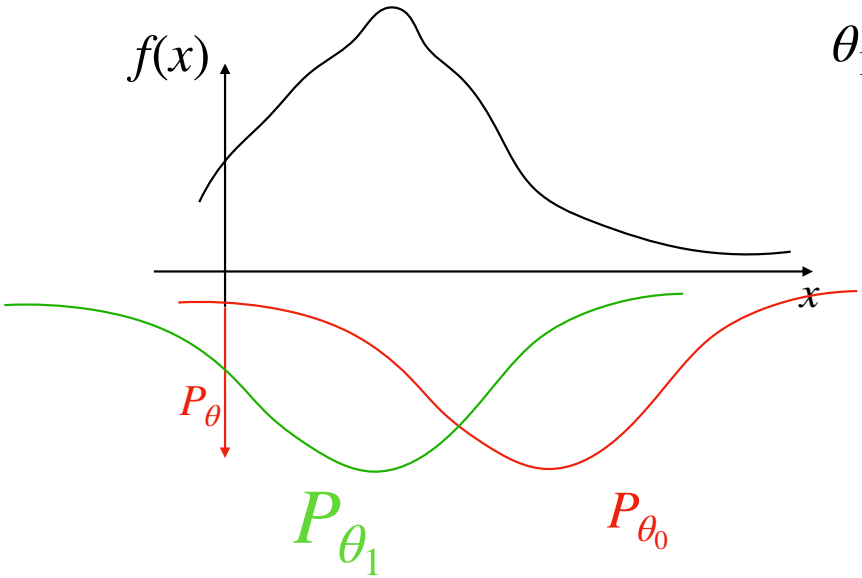


Update distribution (via updating θ) such that P_θ has high probability mass at regions where $f(x)$ is large

Example and Intuition

$$\nabla_\theta J(\theta)\,|_{\theta=\theta_0} = \mathbb{E}_{x\sim P_{\theta_0}} \nabla_\theta \ln P_{\theta_0}(x) \cdot f(x)$$

$$\theta_1 = \theta_0 + \eta \nabla_\theta J(\theta_0)$$



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Using same idea, now let's move on to RL…

Outline for today

1. Recap on Gradient descent and stochastic gradient descent

2. Warm up: computing gradient using importance weighting

3. Policy Gradient formulations

Policy Gradient: Examples of Policy Parameterization (discrete actions)

Recall that we consider parameterized policy $\pi_\theta(\cdot \mid s) \in \Delta(A), \forall s$

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$$\theta_{s,a} \in \mathbb{R}, \forall s, a \in S \times A$$

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Derivation of Policy Gradient: REINFORCE

$$\tau = \{s_0, a_0, s_1, a_1, \ldots\}$$

$P_\theta(\tau)$ = Prob of traj τ under π_θ

$$\rho_\theta(\tau) = \mu(s_0)\pi_\theta(a_0 \mid s_0)P(s_1 \mid s_0, a_0)\pi_\theta(a_1 \mid s_1)\ldots$$

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Adjust policy such that larger reward traj has higher likelihood

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Summary so far for Policy Gradients

We derived the most basic PG formulation:

$$\nabla J(\theta) = \mathbb{E}_{\tau \sim \rho_\theta(\tau)} \left[\left(\sum_{h=0}^{\infty} \nabla_\theta \ln \pi_\theta(a_h \mid s_h) \right) R(\tau) \right]$$

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Increase the likelihood of sampling an trajectory with high total reward

Obtaining a sample $\widetilde{\nabla}_\theta J(\theta)$ for REINFORCE (for this approach)

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- run π_θ on $s_0 \sim \mu$ to get τ
- $\widetilde{\nabla}_\theta J(\theta) = \left(\sum_{h=0}^{\infty} \nabla_\theta \log \pi_\theta(a_h \mid s_h) \right) \left(\sum_{h=0}^{\infty} \gamma^h r_h \right)$

For finite horizon MDP (sometimes used with PG):

$$J(\theta) = \mathbb{E}\left[\sum_{h=0}^{H-1} r(s_h, a_h) \,\middle|\, \mu_0, \pi_\theta\right]$$

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Exercise: Show this simplified version is equivalent to REINFORCE

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Summary for today

1. Importance Weighting (the likelihood ratio method)
2. The Policy Gradient:
   REINFORCE (a direct application of the likelihood ratio method)

$$\nabla J(\theta) = \mathbb{E}_{\tau \sim \rho_\theta(\tau)} \left[\left(\sum_{h=0}^{\infty} \nabla_\theta \ln \pi_\theta(a_h \mid s_h) \right) R(\tau) \right]$$

3. SGAscent With unbiased estimate of $\nabla_\theta J(\theta)$, SGA(hopefully) converges to a local optimal policy.

1-minute feedback form: https://bit.ly/3RHtlxy