“Convergence”
&
Trust Region Policy Optimization

Lucas Janson and Sham Kakade

CS/Stat 184: Introduction to Reinforcement Learning
Fall 2022

Today

- **Next class: embedded ethics (from Jenna Donohue, a postdoc in the Philosophy dept)**
  - **Course Plan: consider different ethical implications of different possible utility functions for a (fictional) RL algorithm that was setting dynamic prices for rides.**
  - **Please come to the next class. (There will be a discussion.)**
  - **Please do the assigned reading (John Rawls) in advance.**

- **Recap++**

- Today:
  1. Convergence of Fitted Policy Iteration
  2. Trust Region Policy Optimization

Recap + Examples

Is there an iterative version of Policy Evaluation?

(that is faster, but approximate?)

Algorithm (Iterative PE):

1. Initialization: $V^0 : \|V^0\|_\infty \in \left[0, \frac{1}{1-\gamma}\right]$
2. Iterate until convergence: $V^{k+1} \leftarrow R + \gamma P V^k$

   Equivalently,

$$\forall s,\ V^{k+1}(s) = r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(s,\pi(s))} V^k(s')$$

$$\vec{V}^{k+1} = \underset{\vec{V}}{\operatorname{argmin}} \|\vec{V} - (R + \gamma P V^k)\|^2$$

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   1. Sample a transition $(s_h, r_h, s_{h+1},) \in D$ and update:

$$V_{k+1}(s_h) = V_k(s_h) - \eta_k\Big(V_k(s_h) - \big(r_h + V_k(s_{h+1})\big)\Big)$$

want this diff in $\mathbb{E}$ to be 0.

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3. Return the function V_K as an estimate of V^π

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Fit $V^{\pi}(s)$ using the iterative policy evaluation alg.

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   1. Construct an *empirical loss function*:

$$L_k(w) = \frac{1}{NH} \sum_{i=1}^{N} \sum_{(s_h, r_h, s_{h+1}) \in \tau_i} \left(f_w(s_h) - \left(r_h + f_{w_k}(s_{h+1}) \right) \right)^2$$

   2. Update with either:
      full minimization:

$$w_{k+1} \approx \arg\min_w L_k(w)$$

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      TD learning: (one step of SGD)

$$w_{k+1} = w_k - \eta_k \widetilde{\nabla} L_k(w_k)$$

stochastic estimate of $\nabla L_k(w_{ik})$

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Fitted Dynamic Programming Methods for learning $Q^\star$ and $\pi^\star$

Policy Iteration (PI)

- Initialization: choose a policy $\pi^0 : S \mapsto A$
- For $k = 0,1,\ldots$
  1. **Policy Evaluation**: compute $Q^{\pi^k}(s,a)$
  2. **Policy Improvement**: set

$$\pi^{k+1}(s) := \arg\max_a Q^{\pi^k}(s,a)$$

Fitted Policy Iteration:
(aka Approximate Policy Iteration API)

1. Initialize staring policy π_0, samples size M
2. For k = 0, … :
   1. [Q-Evaluation Subroutine]

      Using N sampled trajectories, $\tau_1, \ldots \tau_N \sim \rho_{\pi_k}$, try to learn a $\widetilde{Q}$

      $\widetilde{Q}_k(s,a) \approx Q_h^{\pi_k}(s,a) \simeq f_w(s,a,h)$
   2. Policy Update

      $\pi_{k+1}(s) := \arg\max_a \widetilde{Q}^{\pi_k}(s,a)$
3. Return $\widetilde{Q}_K$ and π_K as an estimate of $Q^\star$ and $\pi^\star$

$s \leftarrow (s,h)$

Alternative Version: Bellman Operator $\mathcal{T}$ on Q

(HW2 Q2 is the Q-version of the Bellman Equations)

- Given a function $Q : S \times A \mapsto \mathbb{R}$, define $\mathcal{T}Q : S \times A \mapsto \mathbb{R}$ as

$$\left(\mathcal{T}Q\right)(s,a) := r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a' \in A} Q(s',a')$$

- (Bellman equations for Q)
  Q is equal to $Q^\star$ if and only if $\mathcal{T}Q = Q$.

Q-Value Iteration Algorithm:

1. Initialization: $Q^0 : \|Q^0\|_\infty \in \left[0, \frac{1}{1-\gamma}\right]$
2. Iterate until convergence: $Q_{k+1} \leftarrow \mathcal{T} Q_k$

$$Q_{k+1}(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a' \in A} Q_k(s',a')$$

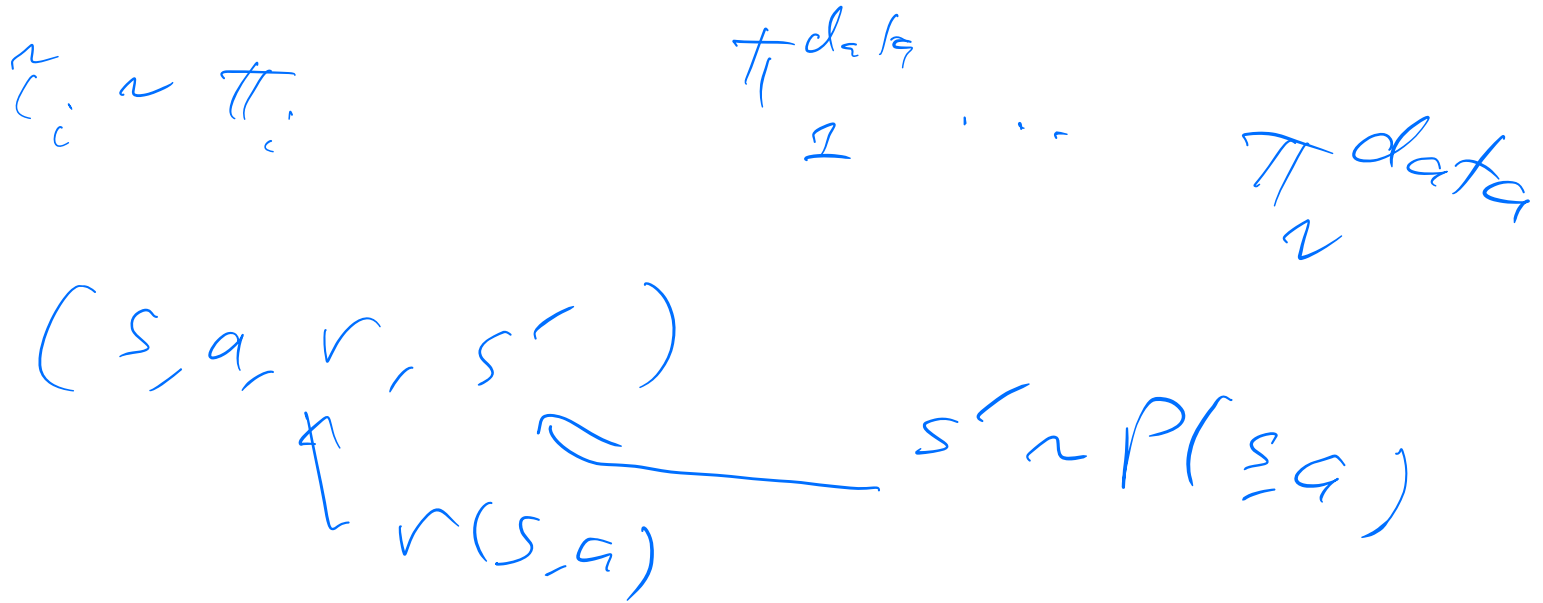
$$\vec{Q}_{k+1} = \min_{\vec{Q}} \| \vec{Q} - (\vec{R} + \gamma P^{\text{"max"}} \vec{Q}_k) \|^2$$

The Offline Learning Setting:

We don't know the MDP and our data collection is under some fixed distribution.

The Finite Horizon, Offline Learning Setting:

- We have N trajectories $\tau_1, \ldots \tau_N \sim \rho_{\pi_{data}}$
- π_{data} is often referred to as our data collection policy.



Q-Learning for "tabular" case

[Iterative Policy Eval Subroutine/TD]

input: **offline dataset** $\tau_1, \ldots \tau_N \sim \rho_{\pi_{data}}$

1. For k = $0, \ldots, K$:
    1. Sample a transition $(s_h, a_h, r_h, s_{h+1}) \in D$ and update:

$$Q_{k+1}(s_h, a_h) = Q_k(s_h, a_h) - \eta_k \Big(Q_k(s_h, a_h) - \big(r_h + \max_{a'} Q_k(s_{h+1}, a')\big) \Big)$$

2. Return the function Q_K as an estimate of $Q^\star$

Fitted Q-Iteration

input: **offline dataset** $\tau_1, \ldots \tau_N \sim \rho_{\pi_{data}}$, init w_0

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   2. Update with either:
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2. Return the function $f_{\widetilde{w}_K}$ as an estimate of $Q^\star$

Today:

“Convergence” & Trust Region Policy Optimization

Outline:

1. Convergence of Fitted Policy Iteration
    1. “Tabular” case
    2. Fitted case
2. Trust Region Policy Optimization
    1. Quick intro on KL-divergence
    2. TRPO formulation

Sample Based Policy Iteration in the Tabular Case:

(the easiest case to think about fitted Policy Iteration)

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      2. Set $\widetilde{Q}_k(s, a)$ as the empirical average of the cumulative reward on these trajectories.

         (i.e. $\widetilde{Q}_k(s, a)$ is unbiased sample of $Q_h^{\pi_k}(s, a)$, with N samples)

if we had a simulator for

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no $|S|$ dependence

M

$M \approx O\left(\frac{\text{poly}(H)}{\epsilon^2}\right)$

[Theorem] Using polynomial many total samples and polynomial computation time (in $|S|, |A|, H, 1/\epsilon$), we have that $\|\widetilde{Q}_K - Q^\star\|_\infty \le \epsilon$ and $\|Q^{\pi_K} - Q^\star\|_\infty \le \epsilon$.

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First: let’s summarize a few things about Supervised Learning

Recap on Supervised Learning: Classification

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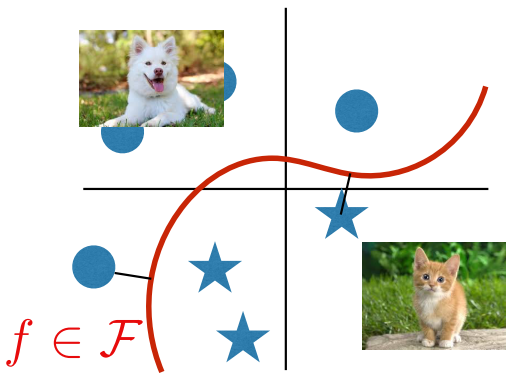

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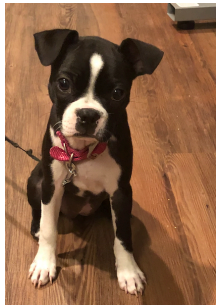
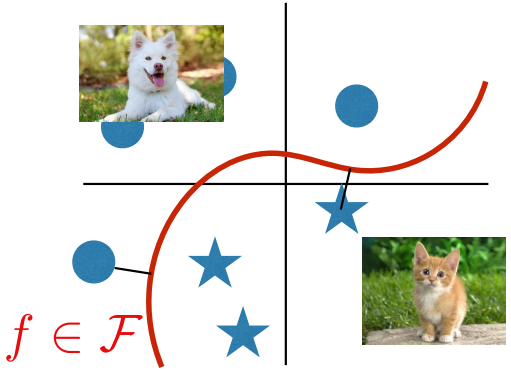
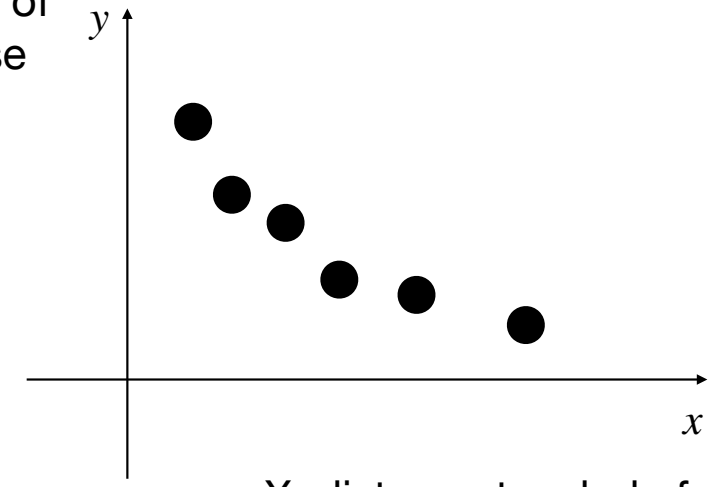
( ,cat) ( ,cat) ( ,dog)



Using function approximator, we are able to predict on cats/dogs that we **never see before** (i.e., we **generalize**)

Recap on Supervised Learning: Regression

Y: value of a house



X: distance to whole foods

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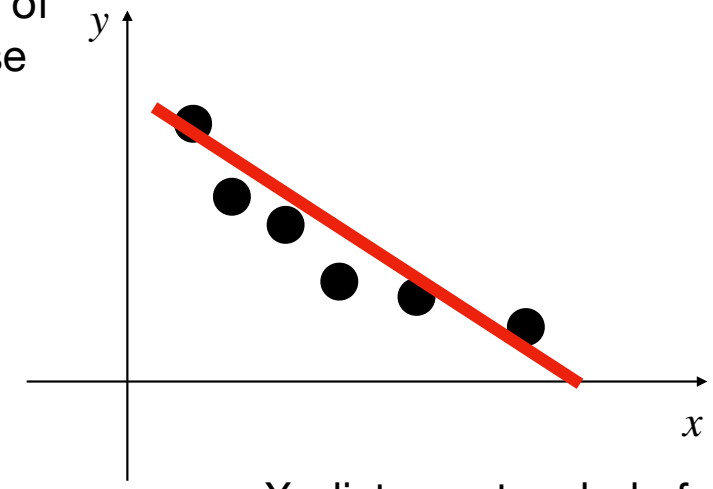

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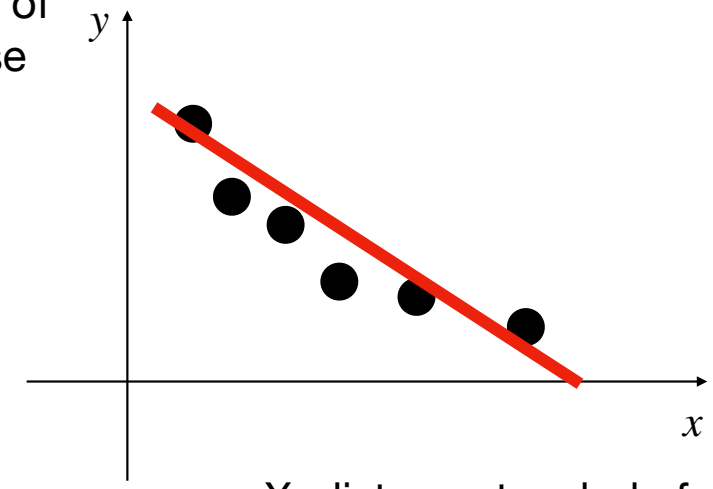

X: distance to whole foods

Using function approximation, we are able to predict on the value of some house not from the training data

Recap on Supervised Learning: regression

We have a data distribution $\mathscr{D}$, $x_i \sim \mathscr{D}$, $y_i = f^\star(x_i) + \epsilon_i$, where noise $\mathbb{E}[\epsilon_i] = 0, |\epsilon_i| \leq c$

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Q: quality of ERM $\hat{f}$?

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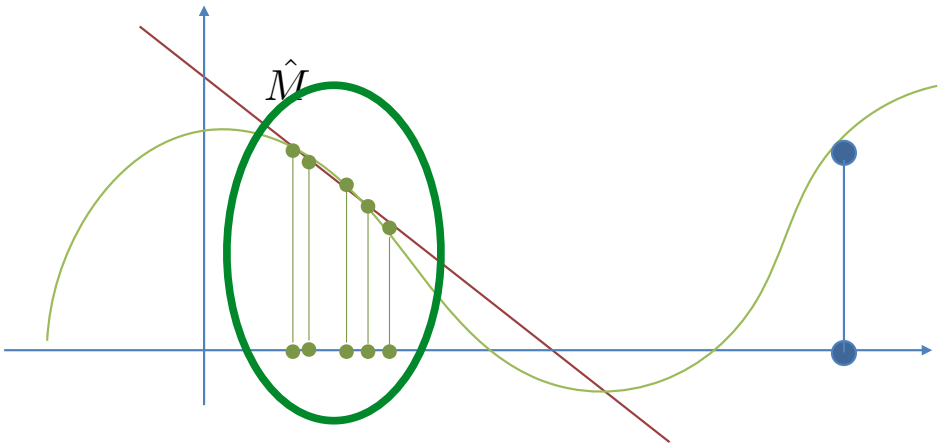
Assume $f^\star \in \mathscr{F}$ (this is called realizability), we can expect:

$$\mathbb{E}_{x \sim \mathscr{D}} \left(\hat{f}(x) - f^\star(x)\right)^2 \leq \delta$$

δ is some function of the complexity of "$\mathscr{F}$"

Supervise Learning can fail if there is train-test distribution mismatch

However, for some $\mathscr{D}' \neq \mathscr{D}$, $\mathbb{E}_{x \sim \mathscr{D}'} (f(x) - f^{\star}(x))^2$ might be arbitrarily large



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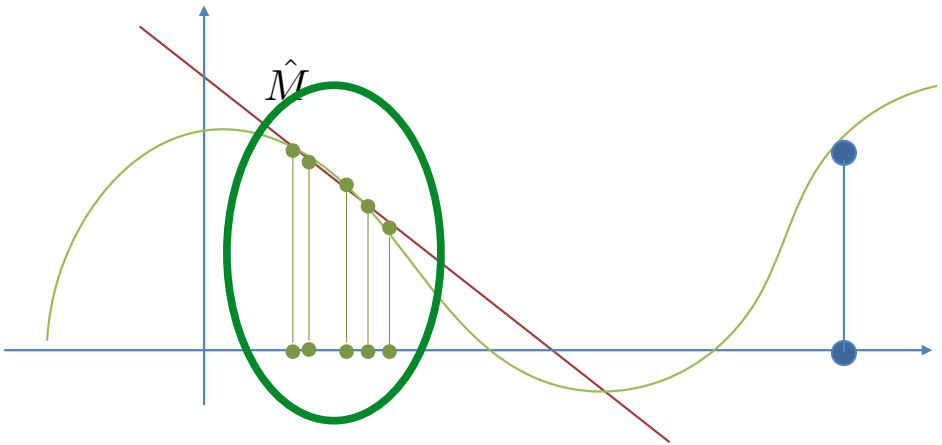

Deeper neural nets and larger datasets are typically
not enough to address “distribution shift”

Back to RL

Fitted Policy Improvement Guarantees

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- For all k, suppose that:

$$E_{\tau\sim\rho_{\pi_k}}\left[\sum_{h=1}^{H}\left(\widetilde{Q}_k(s_h,a_h)-Q_h^{\pi_k}(s_h,a_h)\right)^2\right]\leq\delta, \text{ and } \max_{s,a}\left|\widetilde{Q}_k(s_h,a_h)-Q_h^{\pi_k}(s_h,a_h)\right|\leq\delta_\infty$$

Fitted Policy Improvement Guarantees

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$$E_{\tau\sim\rho_{\pi_k}}\left[\sum_{h=1}^{H}\left(\widetilde{Q}_k(s_h,a_h)-Q_h^{\pi_k}(s_h,a_h)\right)^2\right]\leq\delta, \text{ and } \max_{s,a}\left|\widetilde{Q}_k(s_h,a_h)-Q_h^{\pi_k}(s_h,a_h)\right|\leq\delta_\infty$$

- δ: the average case supervised learning error (reasonable to expect this can be made small)

  δ_∞: the worse case error (often unreasonable to expect to be small)

Fitted Policy Improvement Guarantees

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$$E_{\tau\sim\rho_{\pi_k}}\left[\sum_{h=1}^{H}\left(\widetilde{Q}_k(s_h,a_h)-Q_h^{\pi_k}(s_h,a_h)\right)^2\right]\leq\delta, \text{ and } \max_{s,a}\left|\widetilde{Q}_k(s_h,a_h)-Q_h^{\pi_k}(s_h,a_h)\right|\leq\delta_\infty$$

- δ: the average case supervised learning error (reasonable to expect this can be made small)

  δ_∞: the worse case error (often unreasonable to expect to be small)

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- One step performance degradation is bounded by the worst case error:

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- For large enough K, final performance also governed by the worst case error:

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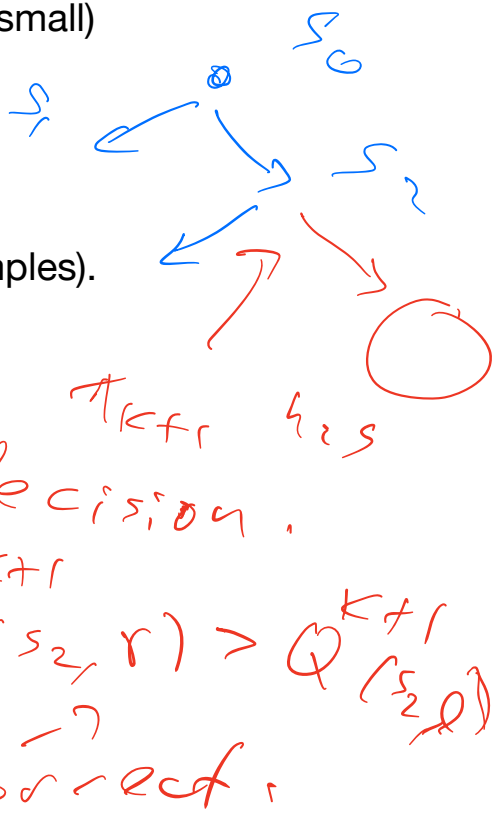

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- (Intuition) If it somehow turns out that, for all iterations k, the density under the next policy, uniformly does not differ from that of previous policy, i.e. that

$$\max_{s,a,h}\left(\frac{\Pr(s_h=s,a_h=a\,|\,\pi_{k+1})}{\Pr(s_h=s,a_h=a\,|\,\pi_k)}\right)\leq C_\infty$$

then we can bound our sub-optimality by the average case error:

$$Q^{\pi_K}(s)\geq Q^\star(s)-2H^2\cdot C_\infty\cdot\delta$$

along the path of the algo.

Outline:

1. Convergence of Fitted Policy Iteration
    1. “Tabular” case
    2. Fitted case
2. Trust Region Policy Optimization
    1. Quick intro on KL-divergence
    2. TRPO formulation

KL-divergence: measures the distance between two distributions

Given two distributions P & Q, where $P \in \Delta(X), Q \in \Delta(X)$,
KL Divergence is defined as:

$$KL(P \mid Q) = \mathbb{E}_{x \sim P}\left[\ln \frac{P(x)}{Q(x)}\right]$$

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If $P = \mathcal{N}(\mu_1, \sigma^2 I), Q = \mathcal{N}(\mu_2, \sigma^2 I)$, then $KL(P \mid Q) = \|\mu_1 - \mu_2\|_2^2 / \sigma^2$

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Fact:

$KL(P \,|\, Q) \geq 0$, and being 0 if and only if $P = Q$

Outline:

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2. Trust Region Policy Optimization
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An “idealized” trust region formulation for policy update: (back to direct policy optimization)

At iteration t, with π_{θ_t} at hand, we compute θ_{t+1} as follows:

$$\max_{\pi_\theta} J(\theta) - J(\theta_t)$$

$$\text{s.t., } KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \leq \delta$$

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How we can actually do the optimization here?
After all, we don't even know the analytical form of trajectory likelihood…

Summary:

1. Convergence of Fitted Policy Iteration
2. Trust Region Policy Optimization
    1. Quick intro on KL-divergence
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1-minute feedback form: https://bit.ly/3RHtlxy