fitted Value function methods & fitted Dynamic Programming

Lucas Janson and Sham Kakade

CS/Stat 184: Introduction to Reinforcement Learning

Fall 2022

Today

- Recap + Overview of PG

- Today:
  1. Fitted Policy Evaluation
  2. Fitted Dynamic Programming Methods
     1. Fitted Policy Evaluation
     2. Fitted Q-Value Iteration

Recap + Overview of PG

Recap: Policy Parameterization

Recall that we consider parameterized policy $\pi_\theta(\cdot \mid s) \in \Delta(A), \forall s$

1. Softmax linear Policy

Feature vector $\phi(s, a) \in \mathbb{R}^d$, and parameter $\theta \in \mathbb{R}^d$

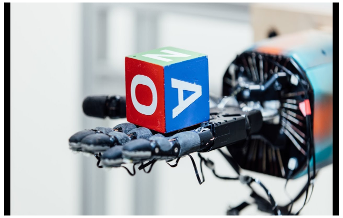
$$\pi_\theta(a \mid s) = \frac{\exp(\theta^\top \phi(s, a))}{\sum_{a'} \exp(\theta^\top \phi(s, a'))}$$

2. Neural Policy:

Neural network $f_\theta : S \times A \mapsto \mathbb{R}$

$$\pi_\theta(a \mid s) = \frac{\exp(f_\theta(s, a))}{\sum_{a'} \exp(f_\theta(s, a'))}$$

Policy Parameterization Example for “Controls”



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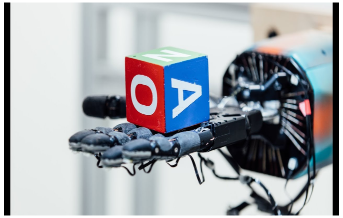

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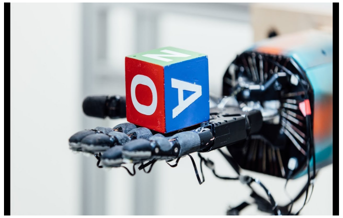

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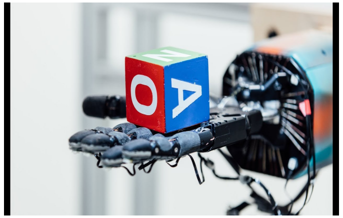

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- Neural network $g_\theta : S \mapsto \mathbb{R}^k$
- Parameters: $\theta \in R^d$, $\sigma \in R^+$
- Policy: sample action from a (multivariate) Normal with mean $g_\theta(s)$ and variance $\sigma^2 I$, i.e.

$$\pi_{\theta,\sigma}(a \mid s) = \mathcal{N}(g_\theta(s), \sigma^2 I)$$

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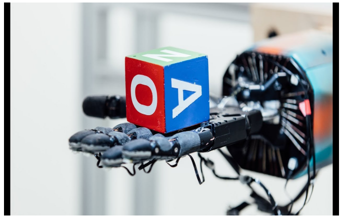

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- Implicitly, this is the same functional form as 2:

$$f_{\theta,\sigma}(s,a) = \frac{\|a - g_\theta(s)\|^2}{2\sigma^2 k}$$

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$$A_h^\pi(s,a) = Q_h^\pi(s,a) - V_h^\pi(s)$$

- We have that:

$$E_{a\sim\pi(\cdot|s)}\left[A_h^\pi(s,a)\,\middle|\,s,h\right] = \sum_a \pi(a\,|\,s)A_h^\pi(s,a) = \;0$$

The PG: baseline and advantage versions

$$\nabla J(\theta) = \mathbb{E}_{\tau \sim \rho_\theta(\tau)} \left[\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h | s_h) \left(\left(\sum_{t=h}^{H-1} r_t \right) - b_h(s_h) \right) \right]$$

$$= \mathbb{E}_{\tau \sim \rho_\theta(\tau)} \left[\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h | s_h) \left(Q_h^{\pi_\theta}(s_h, a_h) - b_h(s_h) \right) \right]$$

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nicest for sampling

- The second step follows by choosing $b_h(s) = V_h^\pi(s)$.
- The most common approach is to use $b_h(s)$ to approximate $V_h^\pi(s)$.
- The REINFORCE version is not used in practice.

PG for the (softmax) linear policies

- We can simplify this to:

$$\nabla J(\theta) = \mathbb{E}_{\tau \sim \rho_\theta(\tau)} \left[\sum_{h=0}^{H-1} A_h^{\pi_\theta}(s_h, a_h) \phi(s_h, a_h) \right]$$

“Review”

- For a random variable $y \in R$, what is:

$$\arg\min_c E_{y \sim D}[(c-y)^2] = ??$$

- Now let us look at the “function” case where we have a distribution over (x, y) pairs

$$f^\star = \arg\min_{f \in \mathscr{F}} E_{(x,y) \sim D}[(f(x)-y)^2]$$

(where $\mathscr{F}$ is the class of all possible functions)

What is $f^\star(x) = ??$ $f(x) \approx E\{y/x\}$

Recap + Overview of PG

A PG procedure:

(this is sometimes referred to as actor-critic approach)

1. Initialize θ_0, samples sizes M,N, parameters: $\eta_1, \eta_2, \ldots$
2. For t = 0, … :
   1. [Policy Eval Subroutine]

      Using N sampled trajectories, $\tau_1, \ldots \tau_N \sim \rho_{\theta_t}$, try to learn a $\widetilde{b}$ s.t.

      $\widetilde{b}(s) \approx V_h^{\pi_{\theta_t}}(s)$
   2. [Mini-Batch PG Update]

      Init $g = 0$ and do M times:

      Obtain a trajectory $\tau \sim \rho_{\theta_t}$

      Set $g = g + \sum_{h=0}^{H-1} \nabla \ln \pi_{\theta_t}(a_h | s_h)\big(R_h(\tau) - \tilde{b}(s_h)\big)$

      Set $\widetilde{\nabla}_\theta J(\theta_t) := \frac{1}{M} g$

   3. Update: $\theta_{t+1} = \theta_t + \eta_t \widetilde{\nabla}_\theta J(\theta_t)$

$R_h(\tau) = \sum_{t=h}^{H-1} r_t$

Baseline/Value Function Parameterizations

Now let us consider parameterized classes of functions $\mathscr{F}$, where for each $f \in \mathscr{F}$, $f : S \to R$

1. Linear Functions

Feature vector $\psi(s) \in \mathbb{R}^k$, and parameter $w \in \mathbb{R}^k$

$$f_w(s) = w^\top \psi(s)$$

2. Neural Policy:

Neural network $f_w : S \mapsto \mathbb{R}$

Let's assume the current time in the episode is contained in the state. $s \leftarrow (s, h)$
(e.g. you can always add the time into the "list" that specifies the state).

Example [Policy Eval Subroutine]:
Directly fit unbiased estimates of $V^\pi(s)$

input: policy π, sample size N

1. Sample trajectories $\tau_1, \ldots \tau_N \sim \rho_\pi$

   (each trajectory is of the form $\tau_i = \{s_0, a_0, r_0, \ldots s_{H-1}, a_{H-1}, r_{H-1}, \})$
2. Construct an *empirical loss function*:

$$\widetilde{L}(w) = \frac{1}{N} \sum_{i=1}^{N} \sum_{s_h \in \tau_i} \Big(f_w(s_h) - R_h(\tau_i) \Big)^2$$

$$= \mathbb{E}_{\tau \sim \rho_\pi} \left[\sum_{h=1}^{H-1} \left(f_w(s_h) - R_h(\tau) \right)^2 \right]$$

3. (approximately) find a minimizer

$$\widetilde{w} \approx \arg\min_{w} \widetilde{L}(w)$$

   (often done with SGD)
4. Return the function $\widetilde{b} = f_{\widetilde{w}}$

Today:

Fitted Value Function & Dynamic Programming Methods

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- Option 1 (without appending h):
  - Try to find parameters H parameters $w^0, \ldots w^{H-1}$, with each in $w_i \in R^k$ s.t. $f_{w^h}(s) \approx V_h^\pi(s)$.
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- Option 2 (building just one model):
  - try to learn a single $w \in R^k$ s.t. $f_w(s, h) \approx V_h^\pi(s), \forall h$.
  - Let us assume: $s \leftarrow (s, h)$ (i.e. that h is implicitly contained in s)

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with $\gamma = 1$

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- It is helpful to use with the discounted algorithms work — Iterative PE, Value Iteration, Policy Iteration — because our parameter w should be effective for all h.

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1. Initialization: $V^0 : \|V^0\|_\infty \in \left[0, \frac{1}{1-\gamma}\right]$
2. Iterate until convergence: $V^{t+1} \leftarrow R + \gamma P V^t$

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$f_{w_k} \longleftrightarrow V^k$

This is a "fixed point" algorithm trying to enforce Bellman consistency:

$$\forall s, V^\pi(s) = r(s,\pi(s)) + \gamma \mathbb{E}_{s' \sim P(s,\pi(s))} V^\pi(s')$$

$\gamma = 1$, $s \leftarrow (s,h)$

The Bellman consistency for the finite horizon case:

$$\forall s, V_h^\pi(s) = r(s,\pi(s)) + \mathbb{E}_{s \sim P(s,\pi(s))} V_{h+1}^\pi(s')$$

same consistency conditions

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Fit $V^{\pi}(s)$ using the iterative policy evaluation alg.

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   1. Construct an *empirical loss function*:

$$\widetilde{L}_k(w) = \frac{1}{N}\sum_{i=1}^{N}\sum_{(s_h,r_h,s_{h+1})\in\tau_i}\Big(f_w(s_h) - \big(r_h + f_{w_k}(s_{h+1})\big)\Big)^2$$

$E\,[f_{w_k}(s')]$

$s' \sim P(s_h, a_h)$

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   2. Update:

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Temporal Difference Learning (TD) is an online variant to do the above.

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Direct Policy Optimization:

- With PG, we tried to directly learn a good (parameterized) policy π_θ, for $\theta \in R^d$
  - Learning means we used only sampled trajectories (we didn't assume the MDP is known).
  - Fitted value functions were introduced for variance reduction.

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Fitted Dynamic Programming:

- Can we instead use a learning (fitting) approach to approximate dynamic programming?

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Policy Iteration (PI)

- Initialization: choose a policy $\pi^0 : S \mapsto A$
- For $t = 0, 1, \ldots$
  1. **Policy Evaluation**: compute $Q^{\pi^t}(s, a)$
  2. **Policy Improvement**: set

$$\pi^{t+1}(s) := \arg\max_a Q^{\pi^t}(s, a)$$

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    1. [Q-Evaluation Subroutine]

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   2. Policy Update
      $\pi_{t+1}(s) := \arg\max_a \widetilde{Q}^{\pi_t}(s,a)$

function of π

LQR

$Q_h^{\pi}(s,a) = a^\top M a$

$a^\top g_\theta(s)\, a$

Q-Function Parameterizations

(nothing new at this point)

Now, for our parameterized classes of functions $\mathscr{F}$, we have for $f \in \mathscr{F}, f : S \times A \to R$

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Feature vector $\psi(s, a) \in \mathbb{R}^k$, and parameter $w \in \mathbb{R}^k$

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Example **[Q-Eval Subroutine]**:
Directly fit unbiased estimates of $Q^\pi(s, a)$

$E[R_h(\tau) | s_h, a_h]$

$= Q_h^\pi(s_h, a_h)$

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As with the [V-Eval Subroutine], there is also an iterative (TD) approach to this.

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    1. Fitted Policy Evaluation
    2. Fitted Q-Value Iteration

Alternative Version: Bellman Operator $\mathcal{T}$ on Q

(HW2 Q2 is the Q-version of the Bellman Equations)

- Given a function $Q : S \times A \mapsto \mathbb{R}$, define $\mathcal{T}Q : S \times A \mapsto \mathbb{R}$ as

$$\big(\mathcal{T}Q\big)(s,a) := \underbrace{r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a' \in A} Q(s',a') = Q(s,a)}_{\text{B.E. for } Q.}$$

- (Bellman equations for Q)
Q is equal to $Q^\star$ if and only if $\mathcal{T}Q = Q$.

- V is eq. to V^* iff $\tilde{\mathcal{T}} V = V$

$$\tilde{\mathcal{T}} V(s) = \max_a \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} V(s') \right]$$

Q-Value Iteration Algorithm:

$\pi_k(s) = \arg\max_a Q_k(s,a)$

1. Initialization: $Q^0 : \|Q^0\|_\infty \in \left[0, \frac{1}{1-\gamma}\right]$
2. Iterate until convergence: $Q_{k+1} \leftarrow \mathcal{T} Q_k$

i.e. $\forall s, a$

$$Q_{k+1}(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a' \in A} Q_k(s', a')$$

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- Guarantees of Q-VI: analogous contraction properties to VI.
- What about a fitted version of this algorithm?

Fitted Q-Iteration

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1. Initialize: Q_0
2. For $k = 0,1,\ldots$:
    $\mathcal{T} Q_k$
    1. Approximately try to estimate $\cancel{\mathcal{T} f_{w_k}}$ with samples

$$Q_{k+1}(s,a) \approx r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a' \in A} Q_k(s',a')$$

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What distribution should use to for this fitting??

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The Finite Horizon, Offline Learning Setting:

- We have N trajectories $\tau_1, \ldots \tau_N \sim \rho_{\pi_{data}}$
- π_{data} is often referred to as our data collection policy.

$\pi_{data}(a|s)$

(we $\overset{(1)}{\text{may}}$ or $\underbrace{\overset{(2)}{\text{may not}}}$ know π_{data})

can't do IS

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   1. Construct an *empirical loss function*:

$r_h(s,a) + \mathbb{E}_{s' \sim P(s,a)} \left[\max_{a} Q_k(s', a')\right]$

$$\widetilde{L}_k(w) = \frac{1}{N} \sum_{i=1}^{N} \sum_{(s_h, a_h, r_h, s_{h+1}) \in \tau_i} \left(f_w(s_h, a_h) - \left(r_h + \max_a f_{w_k}(s_{h+1}, a) \right) \right)^2$$

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3. Return the function $f_{\widetilde{w}_K}$ as an estimate of $Q^\star$

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Q-Learning is an online variant to do the above.

Summary:

1. Fitted Policy Evaluation
2. Fitted Dynamic Programming Methods
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   2. Fitted Q-Value Iteration

Next up: fitted DP methods or PG methods?
TRPO and Natural PG connects these two ideas

1-minute feedback form: https://bit.ly/3RHtlxy