fitted Value function methods
&
fitted Dynamic Programming

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Today

- Recap + Overview of PG

- Today:
  1. Fitted Policy Evaluation
  2. Fitted Dynamic Programming Methods
     1. Fitted Policy Evaluation
     2. Fitted Q-Value Iteration

Recap + Overview of PG

Recap: Policy Parameterization

Recall that we consider parameterized policy $\pi_\theta(\cdot \mid s) \in \Delta(A), \forall s$

1. Softmax linear Policy

Feature vector $\phi(s,a) \in \mathbb{R}^d$, and parameter $\theta \in \mathbb{R}^d$

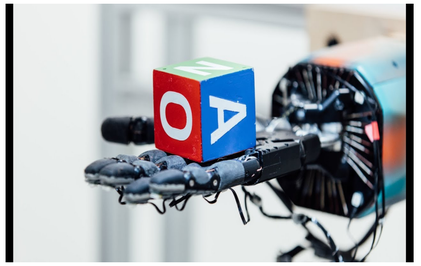
$$\pi_\theta(a \mid s) = \frac{\exp(\theta^\top \phi(s,a))}{\sum_{a'} \exp(\theta^\top \phi(s,a'))}$$

2. Neural Policy:

Neural network
$f_\theta : S \times A \mapsto \mathbb{R}$

$$\pi_\theta(a \mid s) = \frac{\exp(f_\theta(s,a))}{\sum_{a'} \exp(f_\theta(s,a'))}$$

Policy Parameterization Example for “Controls”



Suppose $a \in R^k$, as it might be for a control problem.

2. Neural Policy:

Neural network

$f_\theta : S \times A \mapsto \mathbb{R}$

$$\pi_\theta(a \mid s) = \frac{\exp(f_\theta(s,a))}{\sum_{a'} \exp(f_\theta(s,a'))}$$

3. Example: Neural policy for continuous action case

- Neural network $g_\theta : S \mapsto \mathbb{R}^k$
- Parameters: $\theta \in R^d, \sigma \in R^+$

- Policy: sample action from a (multivariate) Normal with mean $g_\theta(s)$ and variance $\sigma^2 I$, i.e.

$$\pi_{\theta,\sigma}(a \mid s) = \mathcal{N}(g_\theta(s), \sigma^2 I)$$

- Implicitly, this is the same functional form as 2:

$$f_{\theta,\sigma}(s,a) = \frac{\|a - g_\theta(s)\|^2}{2\sigma^2 k}$$

The Advantage Function (finite horizon)

- The Advantage function is defined as:

$$A_h^\pi(s,a) = Q_h^\pi(s,a) - V_h^\pi(s)$$

- We have that:

$$E_{a\sim\pi(\cdot|s)}\left[A_h^\pi(s,a)\,\middle|\,s,h\right] = \sum_a \pi(a\,|\,s)A_h^\pi(s,a) = \;0$$

The PG: baseline and advantage versions

$$\nabla J(\theta) = \mathbb{E}_{\tau\sim\rho_\theta(\tau)}\left[\sum_{h=0}^{H-1}\nabla_\theta \ln \pi_\theta(a_h \,|\, s_h)\left(\left(\sum_{t=h}^{H-1} r_t\right) - b_h(s_h)\right)\right]$$

$$= \mathbb{E}_{\tau\sim\rho_\theta(\tau)}\left[\sum_{h=0}^{H-1}\nabla_\theta \ln \pi_\theta(a_h \,|\, s_h)\left(Q_h^{\pi_\theta}(s_h, a_h) - b_h(s_h)\right)\right]$$

$$= \mathbb{E}_{\tau\sim\rho_\theta(\tau)}\left[\sum_{h=0}^{H-1}\nabla_\theta \ln \pi_\theta(a_h \,|\, s_h) A_h^{\pi_\theta}(s_h, a_h)\right]$$

- The second step follows by choosing $b_h(s) = V_h^\pi(s)$.
- The most common approach is to use $b_h(s)$ to approximate $V_h^\pi(s)$.
- The REINFORCE version is not used in practice.

PG for the (softmax) linear policies

- We can simplify this to:

$$\nabla J(\theta) = \mathbb{E}_{\tau \sim \rho_\theta(\tau)} \left[\sum_{h=0}^{H-1} A_h^{\pi_\theta}(s_h, a_h) \phi(s_h, a_h) \right]$$

“Review”

- For a random variable $y \in R$, what is:

$$\arg\min_{c} E_{y \sim D}[(c - y)^2] = ??$$

- Now let us look at the “function” case where we have a distribution over (x, y) pairs

$$f^\star = \arg\min_{f \in \mathcal{F}} E_{(x,y) \sim D}[(f(x) - y)^2]$$

(where $\mathcal{F}$ is the class of all possible functions)

What is $f^\star(x) = ??$

Recap + Overview of PG

A PG procedure:

(this is sometimes referred to as actor-critic approach)

1. Initialize θ_0, samples sizes M,N, parameters: $\eta_1, \eta_2, \ldots$
2. For t = 0, … :
   1. [Policy Eval Subroutine]

      Using N sampled trajectories, $\tau_1, \ldots \tau_N \sim \rho_{\theta_t}$, try to learn a $\widetilde{b}$ s.t. $\widetilde{b}(s) \approx V_h^{\pi_{\theta_t}}(s)$
   2. [Mini-Batch PG Update]

      Init $g = 0$ and do M times:

      Obtain a trajectory $\tau \sim \rho_{\theta_t}$

      Set $g = g + \sum_{h=0}^{H-1} \nabla \ln \pi_{\theta_t}(a_h \mid s_h)\left(R_h(\tau) - \tilde{b}(s_h)\right)$

      Set $\widetilde{\nabla}_\theta J(\theta_t) := \frac{1}{M} g$
   3. Update: $\theta_{t+1} = \theta_t + \eta_t \widetilde{\nabla}_\theta J(\theta_t)$

Baseline/Value Function Parameterizations

Now let us consider parameterized classes of functions $\mathcal{F}$, where for each $f \in \mathcal{F}$, $f : S \to R$

1. Linear Functions

Feature vector $\psi(s) \in \mathbb{R}^k$, and parameter $w \in \mathbb{R}^k$

$$f_w(s) = w^\top \psi(s)$$

2. Neural Policy:

Neural network $f_w : S \mapsto \mathbb{R}$

Let's assume the current time in the episode is contained in the state. $s \leftarrow (s, h)$
(e.g. you can always add the time into the "list" that specifies the state).

Example [Policy Eval Subroutine]:

Directly fit unbiased estimates of $V^{\pi}(s)$

input: policy π, sample size N

1. Sample trajectories $\tau_1, \ldots \tau_N \sim \rho_{\pi}$

   (each trajectory is of the form $\tau_i = \{s_0, a_0, r_0, \ldots s_{H-1}, a_{H-1}, r_{H-1},\})$

2. Construct an *empirical loss function*:

$$\widetilde{L}(w) = \frac{1}{N} \sum_{i=1}^{N} \sum_{s_h \in \tau_i} \left(f_w(s_h) - R_h(\tau_i) \right)^2$$

3. (approximately) find a minimizer

$$\widetilde{w} \approx \arg\min_{w} \widetilde{L}(w)$$

   (often done with SGD)

4. Return the function $\widetilde{b} = f_{\widetilde{w}}$

Today:

Fitted Value Function & Dynamic Programming Methods

Outline:

1. Fitted Policy Evaluation
2. Fitted Dynamic Programming Methods
    1. Fitted Policy Evaluation
    2. Fitted Q-Value Iteration

Implications of appending the timestep h to the state s

- Option 1 (without appending h):
  - Try to find parameters H parameters $w^0, \ldots w^{H-1}$, with each in $w_i \in R^k$ s.t. $f_{w^h}(s) \approx V_h^\pi(s)$.
  - This is means building H models (or neural nets) so we have $H \cdot k$ parameters.

- Option 2 (building just one model):
  - try to learn a single $w \in R^k$ s.t. $f_w(s, h) \approx V_h^\pi(s),\ \forall h.$
  - Let us assume: $s \leftarrow (s, h)$ (i.e. that h is implicitly contained in s)

- We can implicitly consider this to be an infinite horizon $\gamma = 1$ problem,
  but one which happens to terminate in H steps (i.e. at state $(s, H-1)$ the trajectory ends).

- It is helpful to use with the discounted algorithms work — Iterative PE, Value Iteration, Policy Iteration — because our parameter w should be effective for all h.

Is there an iterative version of Policy Evaluation?

(that is faster, but approximate?)

Algorithm (Iterative PE):

1. Initialization: $V^0 : \|V^0\|_\infty \in \left[0, \frac{1}{1-\gamma}\right]$
2. Iterate until convergence: $V^{k+1} \leftarrow R + \gamma P V^k$

   Equivalently,

$$\forall s, V^{k+1}(s) = r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(s,\pi(s))} V^k(s')$$

This is a "fixed point" algorithm trying to enforce Bellman consistency:

$$\forall s, V^\pi(s) = r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(s,\pi(s))} V^\pi(s')$$

The Bellman consistency for the finite horizon case:

$$\forall s, V_h^\pi(s) = r(s, \pi(s)) + \mathbb{E}_{s \sim P(s,\pi(s))} V_{h+1}^\pi(s')$$

Another [Policy Eval Subroutine]:

Fit $V^{\pi}(s)$ using the iterative policy evaluation alg.

[Iterative Policy Eval Subroutine/TD]

input: policy π, sample size N, init w_0

1. Sample trajectories $\tau_1, \ldots \tau_N \sim \rho_{\pi}$

   (each trajectory is of the form $\tau_i = \{s_0, a_0, r_0, \ldots s_{H-1}, a_{H-1}, r_{H-1},\})$

2. For k = $0,\ldots, K$:

   1. Construct an *empirical loss function*:

$$\widetilde{L}_k(w) = \frac{1}{N} \sum_{i=1}^{N} \sum_{(s_h, r_h, s_{h+1}) \in \tau_i} \left(f_w(s_h) - \left(r_h + f_{w_k}(s_{h+1}) \right) \right)^2$$

   2. Update:

$$w_{k+1} \approx \arg\min_{w} \widetilde{L}_k(w)$$

3. Return the function $\widetilde{b} = f_{w_K}$

Temporal Difference Learning (TD) is an online variant to do the above.

Outline:

1. Fitted Policy Evaluation
2. Fitted Dynamic Programming Methods
    1. Fitted Policy Evaluation
    2. Fitted Q-Value Iteration

This approach leads us to fitted dynamic programming…

Direct Policy Optimization:

- With PG, we tried to directly learn a good (parameterized) policy π_θ, for $\theta \in R^d$
  - Learning means we used only sampled trajectories (we didn't assume the MDP is known).
  - Fitted value functions were introduced for variance reduction.

Fitted Dynamic Programming:

- Can we instead use a learning (fitting) approach to approximate dynamic programming? (where our goal is approximately find $Q^\star$ and $\pi^\star$)

Outline:

1. Fitted Policy Evaluation
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Policy Iteration (PI)

- Initialization: choose a policy $\pi^0 : S \mapsto A$
- For $k = 0,1,\ldots$
  1. **Policy Evaluation**: compute $Q^{\pi^k}(s,a)$
  2. **Policy Improvement**: set

$$\pi^{k+1}(s) := \arg\max_a Q^{\pi^k}(s,a)$$

Fitted Policy Iteration:

1. Initialize staring policy π_0, samples size M
2. For k = 0, … :
    1. [Q-Evaluation Subroutine]

       Using N sampled trajectories, $\tau_1, \ldots \tau_N \sim \rho_{\pi_k}$, try to learn a $\widetilde{b}$

       $\widetilde{Q}^k(s,a) \approx Q_h^{\pi_k}(s,a)$
    2. Policy Update

       $\pi_{k+1}(s) := \arg\max_a \widetilde{Q}^{\pi_k}(s,a)$

Q-Function Parameterizations

(nothing new at this point)

Now, for our parameterized classes of functions $\mathcal{F}$, we have for $f \in \mathcal{F}, f : S \times A \to R$

1. Linear Functions

Feature vector $\psi(s, a) \in \mathbb{R}^k$, and parameter $w \in \mathbb{R}^k$

$$f_w(s, a) = w^\top \psi(s, a)$$

2. Neural Policy:

Neural network

$f_w : S \times A \mapsto \mathbb{R}$

Example [Q-Eval Subroutine]:

Directly fit unbiased estimates of $Q^\pi(s,a)$

input: policy π, sample size N

1. Sample trajectories $\tau_1, \ldots \tau_N \sim \rho_\pi$
2. Construct an *empirical loss function*:

$$\widetilde{L}(w) = \frac{1}{N}\sum_{i=1}^{N} \sum_{(s_h,a_h)\in\tau_i} \left(f_w(s_h,a_h) - R_h(\tau_i)\right)^2$$

3. (approximately) find a minimizer

$$\widetilde{w} \approx \arg\min_w \widetilde{L}(w)$$

   (often done with SGD)

4. Return the function $\widetilde{b} = f_{\widetilde{w}}$

As with the [V-Eval Subroutine], there is also an iterative (TD) approach to this.

Outline:

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Alternative Version: Bellman Operator $\mathcal{T}$ on Q

(HW2 Q2 is the Q-version of the Bellman Equations)

- Given a function $Q : S \times A \mapsto \mathbb{R}$, define $\mathcal{T}Q : S \times A \mapsto \mathbb{R}$ as

$$\big(\mathcal{T}Q\big)(s,a) := r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a' \in A} Q(s',a')$$

- (Bellman equations for Q)
  Q is equal to $Q^\star$ if and only if $\mathcal{T}Q = Q$.

Q-Value Iteration Algorithm:

1. Initialization: $Q^0 : \|Q^0\|_\infty \in \left[0, \frac{1}{1-\gamma}\right]$
2. Iterate until convergence: $Q_{k+1} \leftarrow \mathcal{T} Q_k$

$$Q_{k+1}(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a' \in A} Q_k(s', a')$$

- Guarantees of Q-VI: analogous contraction properties to VI.
- What about a fitted version of this algorithm?

Fitted Q-Iteration

1. Initialize: Q_0
2. For $k = 0,1,\ldots$:
    1. Approximately try to estimate $\mathcal{T}Q_k$ with samples

$$Q_{k+1}(s,a) \approx r(s,a) + \gamma \mathbb{E}_{s' \sim P(s,a)} \max_{a' \in A} Q_k(s',a')$$

What distribution should use to for this fitting??

The Offline Learning Setting:

We don’t know the MDP and our data collection is under some fixed distribution.

The Finite Horizon, Offline Learning Setting:

- We have N trajectories $\tau_1, \ldots \tau_N \sim \rho_{\pi_{data}}$
- π_{data} is often referred to as our data collection policy.

Fitted Q-Iteration

input: **offline dataset** $\tau_1, \ldots \tau_N \sim \rho_{\pi_{data}}$

1. Initialize parameter w_0

   (each trajectory is of the form $\tau_i = \{s_0, a_0, r_0, \ldots s_{H-1}, a_{H-1}, r_{H-1},\})$

2. For $k = 0,1,\ldots K$:

   1. Construct an *empirical loss function*:

$$\widetilde{L}_k(w) = \frac{1}{N} \sum_{i=1}^{N} \sum_{(s_h, a_h, r_h, s_{h+1}) \in \tau_i} \left(f_w(s_h, a_h) - \left(r_h + \max_a f_{w_k}(s_{h+1}, a) \right) \right)^2$$

   2. Update:

$$w_{k+1} \approx \arg\min_w \widetilde{L}_k(w)$$

3. Return the function $f_{\widetilde{w}_K}$ as an estimate of $Q^\star$

Q-Learning is an online variant to do the above.

Summary:

1. Fitted Policy Evaluation
2. Fitted Dynamic Programming Methods
    1. Fitted Policy Evaluation
    2. Fitted Q-Value Iteration

Next up: fitted DP methods or PG methods?
TRPO and Natural PG connects these two ideas

1-minute feedback form: https://bit.ly/3RHtlxy