Policy Gradient Methods: Estimation

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CS/Stat 184: Introduction to Reinforcement Learning

Fall 2023

Today

- ✓ Recap
- Estimation: REINFORCE
- Variance Reduction
  - Other Gradient Expressions
  - Baselines and Advantages
- Examples

Recap

Optimization Objective

- Consider a parameterize class of policies:

$$\{\pi_\theta(a \mid s) \mid \theta \in \mathbb{R}^d\}$$

(why do we make it stochastic?)

- Objective $\max_\theta J(\theta)$, where

$$J(\theta) := E_{s_0 \sim \mu}\left[V^{\pi_\theta}(s_0)\right] = E_{\tau \sim \rho_{\pi_\theta}}\left[\sum_{h=0}^{H-1} r(s_h, a_h)\right]$$

- Policy Gradient Descent:

$$\theta^{k+1} = \theta^k + \eta \nabla J(\theta^k)$$

Example Policy Parameterizations

Recall that we consider parameterized policy $\pi_\theta(\cdot \mid s) \in \Delta(A), \forall s$

1. Softmax linear Policy

Feature vector $\phi(s, a, h) \in \mathbb{R}^d$, and parameter $\theta \in \mathbb{R}^d$

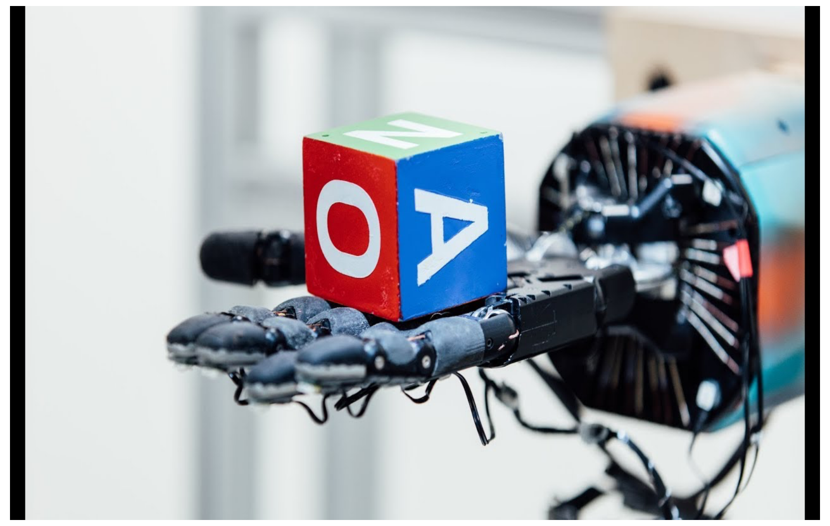
$$\pi_\theta(a \mid s, h) = \frac{\exp(\theta^\top \phi(s, a, h))}{\sum_{a'} \exp(\theta^\top \phi(s, a', h))}$$

2. Neural Policy:

Neural network $f_\theta : S \times A \times [H] \mapsto \mathbb{R}$

$$\pi_\theta(a \mid s, h) = \frac{\exp(f_\theta(s, a, h))}{\sum_{a'} \exp(f_\theta(s, a', h))}$$

Neural Policy Parameterization for “Controls”



Suppose $a \in R^k$, as it might be for a control problem.

3. Gaussian + Linear Model

- Feature vector: $\phi(s,h) \in \mathbb{R}^d$,
- Parameters: $\theta \in \mathbb{R}^{k \times d}$,
  (and maybe $\sigma \in R^+$)

- Policy: sample action from a (multivariate) Normal with mean $\theta \cdot \phi(s,h)$ and variance $\sigma^2 I$, i.e.
$$\pi_{\theta,\sigma}(\cdot \mid s,h) = \mathcal{N}\left(\theta \cdot \phi(s,h), \sigma^2 I\right)$$

4. Gaussian + Neural Model

- Neural network $g_\theta : S \times [H] \mapsto \mathbb{R}^k$
- Parameters: $\theta \in R^d$,
  (and maybe $\sigma \in R^+$)

- Policy: a (multivariate) Normal with mean $g_\theta(s)$ and variance $\sigma^2 I$, i.e.
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- Suppose $J(\theta) = \mathbb{E}_{x \sim P_\theta}\left[f(x)\right] = \sum_x P_\theta(x) f(x)$, and our objective is $\max_\theta J(\theta)$.
- Computing $\nabla_\theta J(\theta)$ exactly may be difficult to compute (due to the sum over x).
  - Can we estimate $\nabla_\theta J(\theta)$?
  - Suppose we can: compute $f(x)$, $P_\theta(x)$, and $\nabla P_\theta(x)$ & sample $x \sim P_\theta$

sample $x_1, \ldots, x_m \sim P_\theta$

$\hat{J} = \frac{1}{m} \sum_{i=1}^{m} f(x_i)$

$G(\theta) = \mathbb{E}_{x \sim D}\left[g_\theta(x)\right]$

$x_1, \ldots, x_m \sim D$

$\hat{G}(\theta) = \frac{1}{m} \sum_{i=1}^{m} g_\theta(x_i)$

$\nabla \hat{G}(\theta) = \frac{1}{m} \sum_{i=1}^{m} \nabla g_\theta(x_i)$

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Proof:

$$\begin{aligned}
\nabla_\theta J(\theta) &= \sum_x \nabla_\theta P_\theta(x) f(x) \\
&= \sum_x P_\theta(x) \frac{\nabla_\theta P_\theta(x)}{P_\theta(x)} f(x) \\
&= \sum_x P_\theta(x) \nabla_\theta \log P_\theta(x) f(x)
\end{aligned}$$

The Likelihood Ratio Method, continued

- We have:

$$\nabla_\theta J(\theta) = \mathbb{E}_{x\sim P_\theta(x)}\left[\nabla_\theta \log P_\theta(x)\, f(x)\right]$$

- An unbiased estimate is given by:

$\widehat{\nabla}_\theta J(\theta) = \nabla_\theta \log P_\theta(x) \cdot f(x)$, where $x \sim P_\theta$

- We can lower variance by draw N i.i.d. samples from P_θ and averaging:

$$\widehat{\nabla}_\theta J(\theta) = \frac{1}{N}\sum_{i=1}^{N} \nabla_\theta \log P_\theta(x_i)\, f(x_i)$$

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- Let $\rho_\theta(\tau)$ be the probability of a trajectory $\tau = \{s_0, a_0, s_1, a_1, \ldots, s_{H-1}, a_{H-1}\}$, i.e.

$$\rho_\theta(\tau) = \mu(s_0)\pi_\theta(a_0 \mid s_0)P(s_1 \mid s_0, a_0)\ldots P(s_{H-1} \mid s_{H-2}, a_{H-2})\pi_\theta(a_{H-1} \mid s_{H-1})$$

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- The REINFORCE Policy Gradient expression:

$$\nabla_\theta J(\theta) := \mathbb{E}_{\tau\sim\rho_\theta}\left[\left(\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\right) R(\tau)\right]$$

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$$= \left(\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \,|\, s_h)\right)$$

Obtaining an Unbiased Gradient Estimate at θ_0

$$\nabla_\theta J(\theta) := \mathbb{E}_{\tau \sim \rho_\theta}\left[\left(\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\right) R(\tau)\right]$$

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We have: $\mathbb{E}[\widetilde{\nabla}_\theta J(\theta_0)] = \nabla_\theta J(\pi_{\theta_0})$

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Other PG formulas
(that are lower variance for sampling)

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Intuition: Change action distribution at h only affects rewards later on…

HW: You will show these simplified version are also valid PG expressions

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On a trajectory τ, define:

$$R_h(\tau) = \sum_{t=h}^{H-1} r_t$$

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Comments:

- We still have unbiased gradient estimates.
- Easy to use a mini-batch algorithm to reduce variance.
- Easy to compute the gradient in "one pass" over the data.

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For any function only of the state, $b_h : S \to R$, we have:

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$E[x] = \mu$

$x \sim D$

$E[\mu - x] = 0$

$\hat{\mu} = \frac{1}{m} \sum_{i=1}^{m} x_i$

$\delta_i = \mu - x_i$

$\hat{\mu} = \frac{1}{m}$

This is (basically) the method of control variates.

$\hat{\mu} = \frac{1}{m} \sum_{i=1}^{m} (x_i + \delta_i) = \mu$

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- To see this, first note:

$$\mathbb{E}_{x\sim P_\theta}\left[\nabla \log P_\theta(x) c\right] = c\sum_x P_\theta(x)\nabla \log P_\theta(x)$$

$$\nabla \log P_\theta(x) = \frac{\nabla P_\theta}{P_\theta}$$

$$= c\sum_x \nabla P_\theta(x) = c\nabla\sum_x P_\theta(x)$$

$$\sum_x P_\theta(x) = 1$$

$$= c\nabla 1 = 0$$

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- The claims follows due to that for any constant c,

$$\mathbb{E}_{x \sim P_\theta}\left[\nabla \log P_\theta(x)\left(f(x) - c\right)\right] = \mathbb{E}_{x \sim P_\theta}\left[\nabla \log P_\theta(x) f(x)\right]$$

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    3. Update: $\theta^{k+1} = \theta^k + \eta^k \widetilde{\nabla}_{\theta} J(\theta^k)$

The Advantage Function (finite horizon)

$$V_h^\pi(s) = \mathbb{E}\left[\sum_{\tau=h}^{H-1} r(s_\tau, a_\tau) \middle| s_h = s\right] \qquad Q_h^\pi(s,a) = \mathbb{E}\left[\sum_{\tau=h}^{H-1} r(s_\tau, a_\tau) \middle| (s_h, a_h) = (s,a)\right]$$

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$\forall a$ $\quad A^{*}_{a}(s, a)$

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- For the discounted case, $A^\pi(s,a) = Q^\pi(s,a) - V^\pi(s)$

Summary:

1. REINFORCE (a direct application of the likelihood ratio method)
2. Variance Reduction: with baselines

Attendance:

bit.ly/3RcTC9T

Feedback:

bit.ly/3RHtlxy