Bandits: Upper Confidence Bound Algorithm

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CS/Stat 184: Introduction to Reinforcement Learning
Fall 2023

Today

- Feedback from last lecture
- Recap
- Confidence intervals for the arms
- Upper Confidence Bound (UCB) algorithm
- UCB regret analysis

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  - Achieve sublinear regret of $\tilde{O}(T^{2/3})$
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- Today: UCB does better than a rate of $T^{2/3}$

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Sample mean of N i.i.d. samples on $[0,1]$ satisfies

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Worked for ETC b/c exploration phase was i.i.d., but in general the rewards from a given arm are *not* i.i.d. due to adaptivity of action selections

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But this is generally FALSE

(unless a_t chosen very simply, like exploration phase of ETC)

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i.e., $r_\tau \mid a_\tau = k$ simply equal to $\tilde{r}_{N_\tau^{(k)}}^{(k)}$, and hence $\hat{\mu}_t^{(k)} = \frac{1}{N_t^{(k)}} \sum_{i=0}^{N_t^{(k)}-1} \tilde{r}_i^{(k)}$

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Hoeffding + union bound over $n \leq t$:

$$\Rightarrow \mathbb{P}\left(\forall n \leq t, |\tilde{\mu}_n^{(k)} - \mu^{(k)}| \leq \sqrt{\ln(2t/\delta)/2n} \right) \geq 1 - \delta$$

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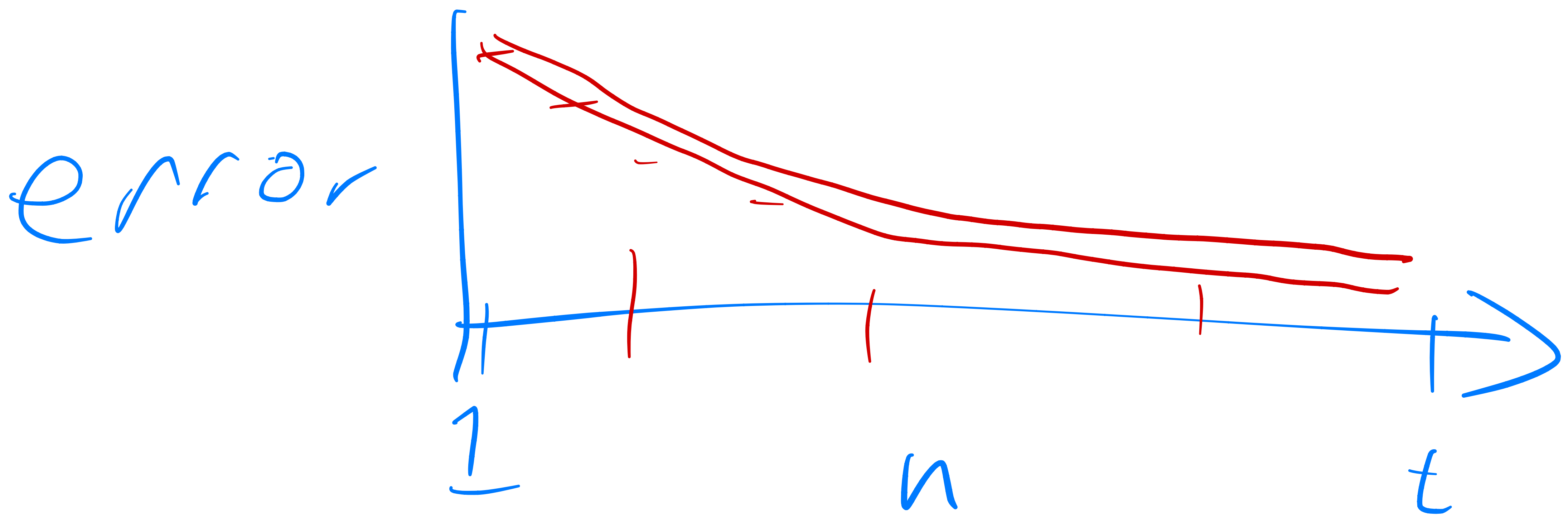

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Summary: to deal with problem of non-i.i.d. rewards that enter into $\hat{\mu}_t^{(k)}$, we used rewards' *conditional* i.i.d. property along with a union bound to get Hoeffding bound that is wider by just a factor of t in the log term

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So we have a valid $(1-\delta)$ confidence interval (CI) for $\mu^{(k)}$ at time t from last equation:

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By same argument made in ETC analysis, union bound over K makes coverage uniform over k:

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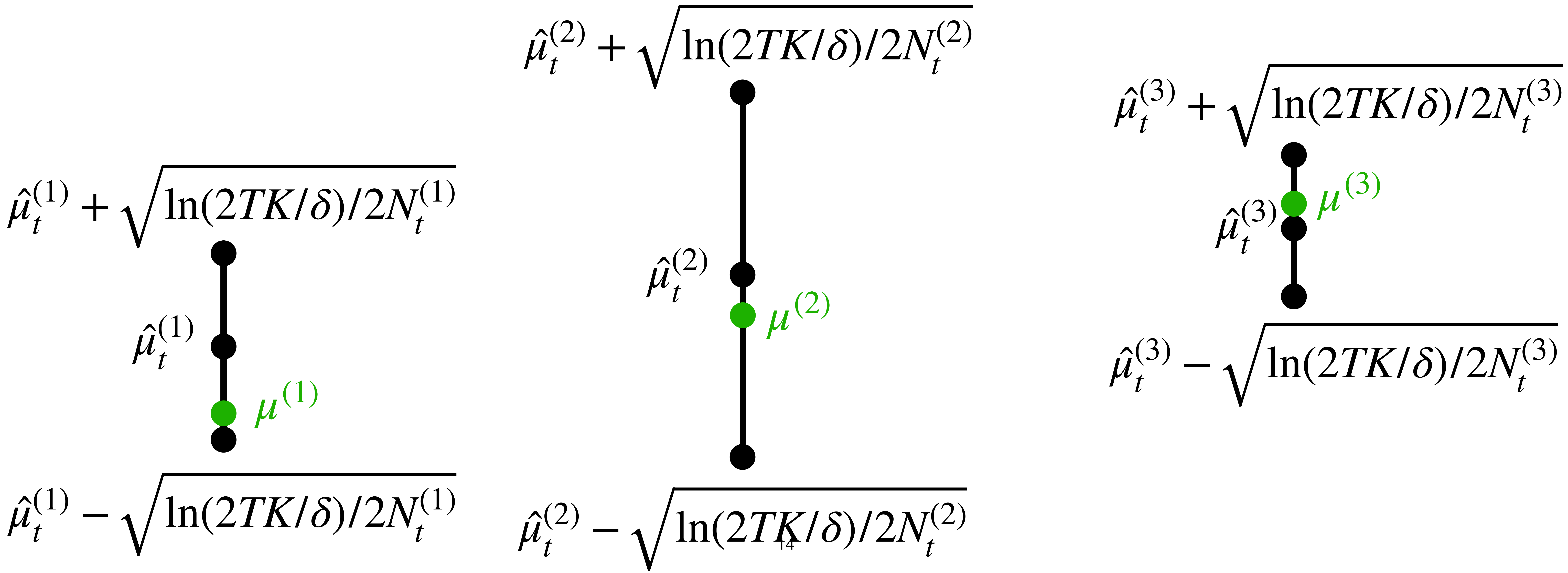

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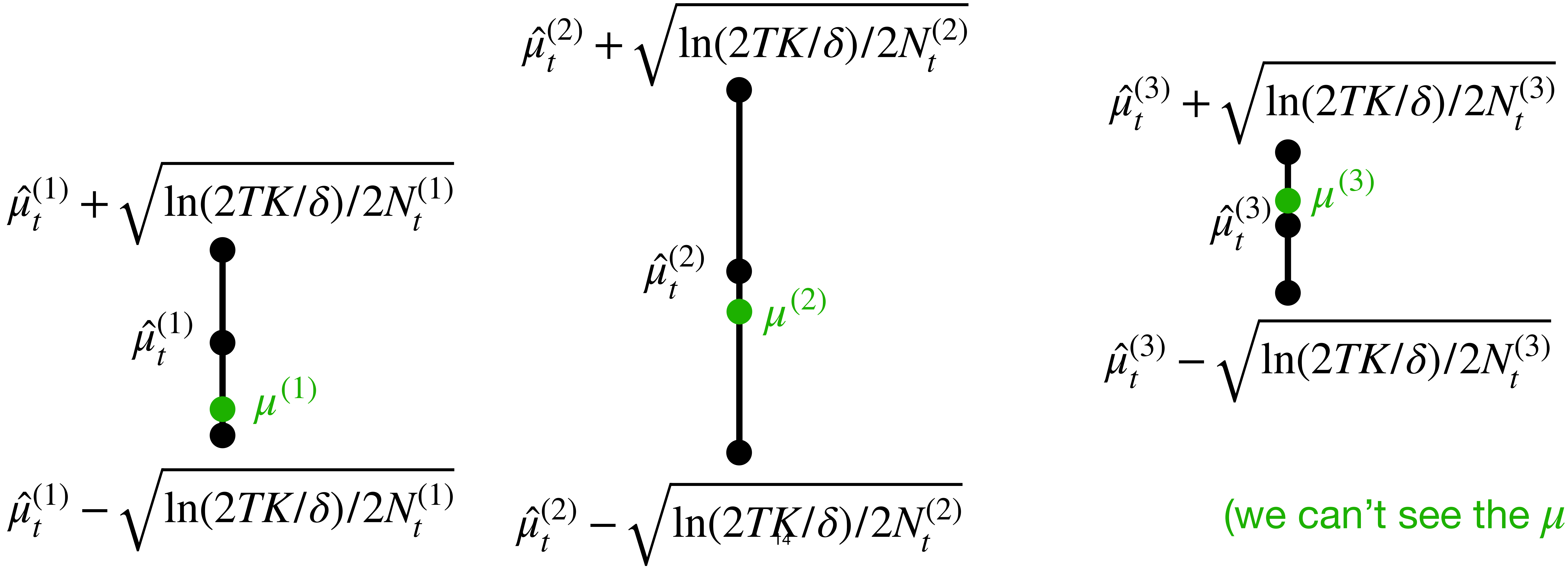

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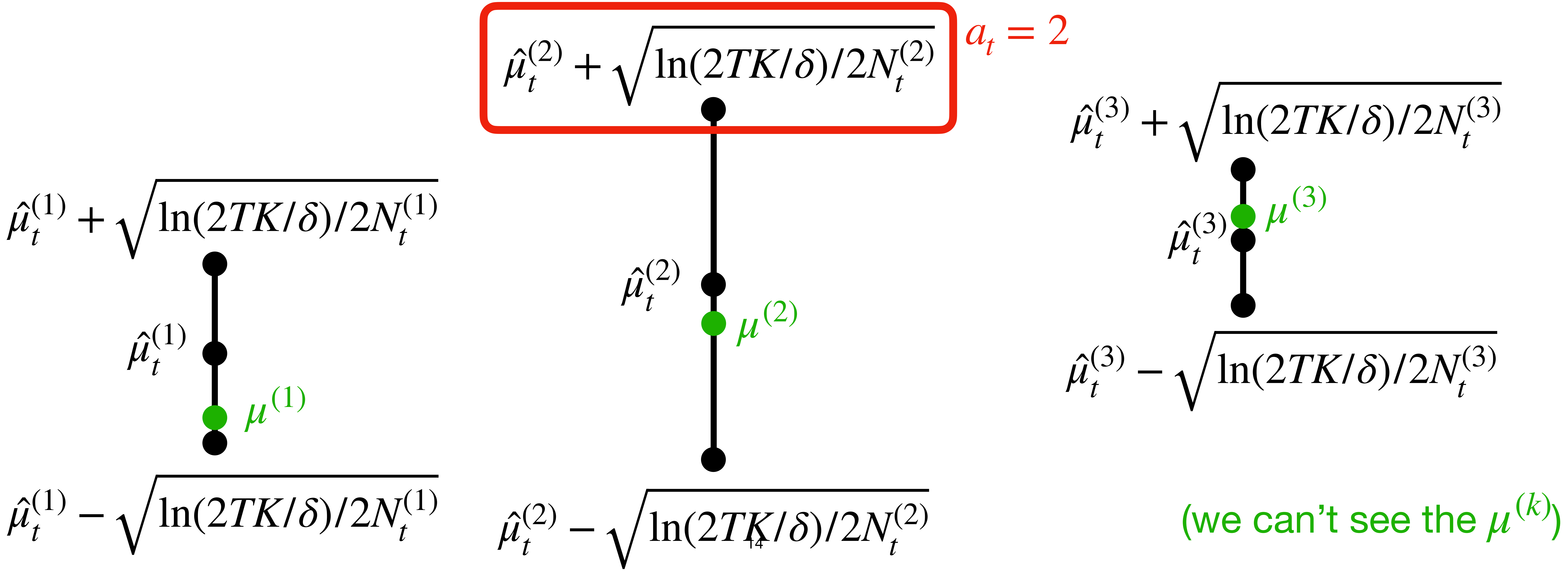

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2. $\hat{\mu}_t^{(k)}$ large, i.e., based on what we've seen so far, arm k is the best (exploitation)

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2. $\hat{\mu}_t^{(k)}$ large, i.e., based on what we've seen so far, arm k is the best (exploitation)

Note that the exploration here is *adaptive*, i.e., focused on most promising arms

Today

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- ✓ Confidence intervals for the arms
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- UCB regret analysis

UCB Regret Analysis Strategy

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2. Bound the sum of those bounds over time steps

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all lines above hold simultaneously for all t w/p $1-\delta$ because of *uniform* Hoeffding

Sum of UCB per-time-step regrets

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$$\sum_{n=1}^{T} \frac{1}{\sqrt{n}} \le 1 + \int_1^T \frac{1}{\sqrt{x}}\, dx = 1 + 2\sqrt{x}\ \Big|_{x=1}^{x=T} = 2\sqrt{T}$$

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Finally, putting it all together, we get:

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In fact, a more sophisticated analysis can get: $\text{Regret}_T = \tilde{O}(\sqrt{KT}) \quad \text{w/p } 1-\delta$

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So far we our theoretical analysis has always considered a fixed algorithm and analyzed it (by deriving a regret upper bound with high probability)

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Short answer: no

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A *lower bound* on the achievable regret

So far we our theoretical analysis has always considered a fixed algorithm and analyzed it (by deriving a regret upper bound with high probability)

To get a lower bound, we would need to consider what regret could be achieved by *any* algorithm, and show it can't be better than some rate

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3. This means that if an arm $\tilde{k}$ is about $1/\sqrt{T}$ away from the best arm $k^\star$, then at no point during the bandit can we confidently tell them apart
4. Thus, we should expect to sample $\tilde{k}$ roughly as often as $k^\star$, which is at best roughly $T/2$ times (if we ignore any other arms)
5. Finally, since the regret incurred each time we pull arm $\tilde{k}$ is $1/\sqrt{T}$, and we pull it $T/2$ times, we get a regret lower bound of $(1/\sqrt{T}) \times T/2 = \Omega(\sqrt{T})$

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Summary:

Upper Confidence Bound (UCB) algorithm:

- Uses uncertainty quantification *inside* algorithm
- Performs adaptive exploration via the principle of *optimism in the face of uncertainty* (OFU)
- Achieves regret of $\tilde{O}(\sqrt{TK})$
- A regret lower-bound exists that says one can't do better than $\Omega(T)$ regret

Attendance:

bit.ly/3RcTC9T

Feedback:

bit.ly/3RHtlxy