Bandits: Bayesian Bandits and Thompson Sampling

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CS/Stat 184: Introduction to Reinforcement Learning
Fall 2023

Today

- Feedback from last lecture
- Recap
- Bayesian bandit
- Thompson sampling

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- UCB uses Optimism in the Face of Uncertainty (OFU) principle and achieves *optimal* rate $\tilde{O}(\sqrt{T})$ of regret
- Theory is nice, but what about in practice?

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One such prior, since all the $\mu^{(k)}$ are bounded between 0 and 1, is the prior that is *Uniform* on the unit hypercube, i.e.,
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Without the Bernoulli assumption, we may need many more dimensions to describe the possible distributions, and hence have to define a much higher-dimensional prior

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   We have no data, and the distribution of the reward distributions is simply given by the prior on the reward parameters $\boldsymbol{\mu}$:
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   ($\mathbb{P}$ will sometimes denote a continuous density instead of a true probability, e.g., for $\boldsymbol{\mu} \sim$ Uniform$([0,1]^K)$, we would write $\mathbb{P}(\boldsymbol{\mu}) = 1_{\{0 \leq \mu^{(k)} \leq 1 \ \forall k\}}$)

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that exactly characterizes our uncertainty about $\boldsymbol{\mu}$.

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Bayesian Bernoulli bandit (cont'd)

Bayesian Bernoulli bandit with uniform prior on $\boldsymbol{\mu}$ gives a running posterior on the mean of each arm k that is $\text{Beta}(1 + \#\{\text{arm } k \text{ successes}\}, 1 + \#\{\text{arm } k \text{ failures}\})$

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which starts at 1/2 and approaches the sample mean of arm k with more pulls.

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$\text{Beta}(\alpha_k, \beta_k)$ has variance (posterior variance ≈ how uncertain we are about $\mu^{(k)}$):

$$\frac{\alpha_k}{\alpha_k + \beta_k} \times \frac{\beta_k}{\alpha_k + \beta_k} \times \frac{1}{\alpha_k + \beta_k + 1}$$

which decreases at a rate of roughly $1/\#\{\text{arm } k \text{ pulls}\}$

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A Bayesian bandit augments the bandit environment we've been working in so far with a prior distribution on the unknown reward distributions; for Bernoulli bandits, the reward distributions are entirely characterized by $\boldsymbol{\mu}$, so prior is: $\pi(\boldsymbol{\mu})$

Bayes rule at time step t gives us a distribution (called the posterior distribution)

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What changes with t is our information about $\boldsymbol{\mu}$, i.e., the posterior distribution, as we collect more and more data by pulling arms via a bandit algorithm

Today

- ✓ Feedback from last lecture
- ✓ Recap
- ✓ Bayesian bandit
- Thompson sampling

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(though computationally, it may not be easy to update the posterior at each time step)

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π can often be chosen "uninformatively" to a default prior such as the uniform, or can encode nuanced prior information/belief about the arms' reward distributions

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Although derived from the Bayesian bandit, Thompson sampling has excellent practical performance across bandit problems, whether or not they are Bayesian!

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There is an *instance-dependent* lower-bound result that says that for any bandit algorithm:

$$\liminf_{T\to\infty} \frac{\mathbb{E}[N_T^{(k)}]}{\ln(T)} \geq \frac{1}{d(\nu^{(k^\star)}, \nu^{(k)})},$$

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(UCB is not, but there are more complicated versions of it that are)

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Thompson sampling doesn't know this, and neither does UCB (although UCB wouldn't happen to make the same mistake in this case).

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Such tuning can improve Thompson sampling's performance even for reasonably large T (the asymptotic optimality of vanilla TS is *very* asymptotic)

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Summary:

- Bayesian bandit adds an additional assumption of prior on reward distributions
- Bayes rule gives exact running uncertainty quantification for any algorithm
- Thompson sampling samples optimal arm from its (posterior) distribution
- Thompson sampling achieves excellent performance in practice

Attendance:

bit.ly/3RcTC9T

Feedback:

bit.ly/3RHtlxy