Fitted Dynamic Programming

Lucas Janson and Sham Kakade CS/Stat 184: Introduction to Reinforcement Learning Fall 2023

- Feedback from last lecture
- Recap
- Neural networks
- Fitted value iteration
- Fitted policy iteration



Feedback from feedback forms

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1. Thank you to everyone who filled out the forms!

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- <u>Linear regression</u> parameterizes f(x) as $x^{\top}\theta$ and can work well when $\mathbb{E}[y | x]$ very smooth, high-dimensional (penalties like ridge/lasso help here), and/or there is a good featurization $\phi(x)$

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- Parameter vector θ concatenates all W's and b's; dim(θ) scales as width \times depth







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- We hope that SGD finds a good one... in practice there are optimization tricks that are like SGD but perform better, e.g., one very popular one is called Adam





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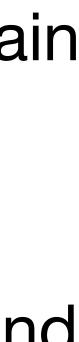




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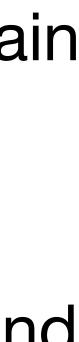


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Practical Neural Networks are very far from "just" ERM







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Recall that Bellman equations state that the optimal value function $V^{\star}(s)$ satisfies: $V^{\star}(s) = \max_{a} \left\{ r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s, a)} \left[V^{\star}(s') \right] \right\}, \quad \forall s$

Recall: Value Iteration Algorithm (infinite horizon, discounted)

1. Initialization: $V^{0}(s) = 0$, 2. For t = 0, ..., T - 1 $V^{t+1}(s) = \max_{a} \left\{ r(s, a) + a \right\}$ 3. Return: $V^{T}(s)$ $\pi(s) = \arg\max_{a} \left\{ r(s, a) + a \right\}$ a

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 - And the VI algorithm is a fixed-point algorithm to find V^{\star} :

$$\begin{aligned} \forall s \\ &+ \gamma \sum_{s' \in S} P(s' \mid s, a) V^t(s') \Big\}, \ \forall s \\ &) + \gamma \mathbb{E}_{s' \sim P(\cdot \mid s, a)} V^T(s') \Big\} \end{aligned}$$

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• Initialize: $V_{H}^{\pi}(s) = 0 \quad \forall s \in S$ For $t = H - 1, \dots 0$, set: • $V_{h}^{\star}(s) = \max_{a} \left[r(s, a) + \mathbb{E} \right]$ • $\pi_{h}^{\star}(s) = \arg\max_{a} \left[r(s, a) - m_{a}^{\star} \right]$

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$$\mathbb{E}_{s' \sim P(\cdot|s,a)} \left[V_{h+1}^{\star}(s') \right], \forall s \in S$$
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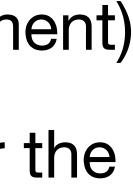
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The above DP algorithm can just be seen as solving SH (Bellman) equations for the SH different values of V(s, h), but doing so in an exact, efficient way via DP

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Recall from HW1, Problem 2, the Bellman equations for Q^* :

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Q-Value Dynamic Programming Algorithm:

$$Q_h^{\star}(s,a) = r(s,a) + \mathbb{E}$$

- 1. Initialization: Q(s, a, H) = 0
- 2. Solve (via dynamic programm

 $Q(s, a, h) = r(s, a) + \mathbb{E}_{s' \sim P(s, a)}$

3. Return:

 $\pi_h(s) = \arg n$

- Recall from HW1, Problem 2, the Bellman equations for Q^* :
 - $\mathbb{E}_{s' \sim P(\cdot|s,a)} \left[\max_{a'} Q_{h+1}^{\star}(s',a') \right]$
- Analogous Q-value DP, with same notational change as previous slide: h as argument

$$\forall s, a \\ \text{ning}: \\ s_{(a)} \left[\max_{a' \in A} Q(s', a', h+1) \right] \quad \forall s, a, h \\ a' \in A \quad \left\{ Q(s, a, h) \right\}$$





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Connection to Supervised Learning $Q(s, a, h) \approx r(s, a) + \mathbb{E}_{s' \sim P(s, a)} \left[\max_{a' \in A} Q(s', a', h+1) \right] \quad \forall s, a, h$

Then we'd be happy if we found a $Q(s_h, a_h, h) = f(x) = \mathbb{E}[y \mid x] = \mathbb{E} |r(x)| = \mathbb{E}[y \mid x] = \mathbb{E}$

$$(s_h, a_h) + \max_{a'} Q(s_{h+1}, a', h+1) \left| s_h, a_h, a_h, a_h' \right|$$



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What are the y and x?

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Note that the RHS can also be written as

$$\mathbb{E}\left[r(s_h, a_h) + \max_{a'} Q(s_{h+1}, a', h+1) \middle| s_h, a_h, h \right]$$

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$$\mathbb{E}\left| r(s_h, a_h) + \max_{a'} Q(s_{h+1}, a', h+1) \right| s_h, a_h, h$$

This suggests that $y = r(s_h, a_h) + \max_{a'} Q(s_{h+1}, a', h+1)$ and $x = (s_h, a_h, h)$ Then we'd be happy if we found a

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Setting that aside for the moment, to fit supervised learning, we'd minimize a leastsquares objective function: $\hat{f}(x) = \arg \min_{f \in \mathcal{F}} \sum_{i=1}^{\infty} (y_i - f(x_i))^2$

BUT, to compute each y, we need to already know Q!



We can convert our data $\tau_1, \ldots, \tau_N \sim \rho_{\pi_{data}}$, into (y, x) pairs; how many? NH

Setting that aside for the moment, to fit supervised learning, we'd minimize a leastsquares objective function: $\hat{f}(x) = \arg \min_{f \in \mathscr{F}} \sum_{i=1}^{\infty} (y_i - f(x_i))^2$

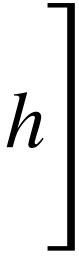
Then if we have enough data, choose a good $\mathcal{F},$ and optimize well,

 $Q(s_h, a_h, h) := \hat{f}(x) \approx \mathbb{E}[y \mid x] = \mathbb{E} | r(y)| = \mathbb{E} |$

BUT, to compute each y, we need to already know Q!

$$(s_h, a_h) + \max_{a'} Q(s_{h+1}, a', h+1) \left| s_h, a_h, a_h, a_h' \right|$$





Fitted (Q-)Value Iteration

To address the circularity problem of not knowing Q for computing the y, we have an algorithmic tool... what is it?



Fitted (Q-)Value Iteration

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- **Fixed point iteration!** Initialize, then at each step, pretend Q is known by plugging in the previous time step's Q to compute the y's, and then use that to get next Q





Fitted (Q-)Value Iteration

Input: offline dataset $\tau_1, \ldots \tau_N \sim \rho_{\pi_{da}}$ 1. Initialize fitted Q function at f_0 2. For $k = \emptyset, 1, ..., K$: $f_k = \arg\min_{f \in \mathscr{F}} \sum_{i=1}^N \sum_{h=1}^{H-1} \left(f(s_h^i, a_h^i, h) - \frac{1}{2} \right) = 0$

3. With f_K as an estimate of Q^{\star} ,

- To address the circularity problem of not knowing Q for computing the y, we have an algorithmic tool... what is it? *Hint*: we used it for another VI algorithm before...
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$$-\left(r(s_{h}^{i}, a_{h}^{i}) + \max_{a} f_{k-1}(s_{h+1}^{i}, a, h+1)\right)\right)^{2}$$

return $\pi_{h}(s) = \arg\max_{a} \left\{f^{K}(s, a, h)\right\}$





Fitted (Q-)Value Iteration

Input: offline dataset $\tau_1, \ldots \tau_N \sim \rho_{\pi_{da}}$

- 1. Initialize fitted Q function at f_0
- 2. For k = 0, 1, ..., K: $f_k = \arg \min_{f \in \mathscr{F}} \sum_{i=1}^{N} \sum_{h=1}^{H-1} \left(f(s_h^i, a_h^i, h) \frac{1}{2} \right) = 0$

3. With f_K as an estimate of Q^{\star} ,

Q-Learning is an online version, i.e., draw new trajectories at each k based on f_k as Q-function

- To address the circularity problem of not knowing Q for computing the y, we have an algorithmic tool... what is it? *Hint*: we used it for another VI algorithm before...
- **Fixed point iteration!** Initialize, then at each step, pretend Q is known by plugging in the previous time step's Q to compute the y's, and then use that to get next Q

$$-\left(r(s_{h}^{i}, a_{h}^{i}) + \max_{a} f_{k-1}(s_{h+1}^{i}, a, h+1)\right)^{2}$$

return $\pi_{h}(s) = \arg\max_{a} \left\{f^{K}(s, a, h)\right\}$







- Recap
- Neural networks
- Fitted value iteration
 - Fitted policy iteration



- Initialization: choose a policy $\pi^0: S \mapsto A$
- For k = 0, 1, ...
 - 1. Policy Evaluation: Solve (via dynamic programming):

 $Q^{\pi^{k}}(s, a, h) = r(s, a) + \mathbb{E}_{s' \sim P(\cdot|s, a)} \left[Q^{\pi^{k}}(s', \pi^{k}(a), h+1) \right]$ $\forall s, a, h$ 2. Policy Improvement: set $\pi_h^{k+1}(s) := \arg \max Q^{\pi^k}(s, a, h)$



Initialization: choose a policy π⁰ : S → A
For k = 0,1,...
1. Policy Evaluation: Solve (via dynamic Q^{π^k}(s, a, h) = r(s, a) + E_{s'~P(·|s,a)} [Q
2. Policy Improvement: set π^{k+1}_h(s) :=

Again: what if we're in full RL setting where we can't just evaluate expectations?

dynamic programming):

$$P(\cdot|s,a) \begin{bmatrix} Q^{\pi^{k}}(s', \pi^{k}(a), h+1) \end{bmatrix} \quad \forall s, a, h \\ \forall s, a, h \\ \exists s, a, h \end{bmatrix}$$



• Initialization: choose a policy $\pi^0 : S \mapsto A$ • For k = 0, 1, ...1. Policy Evaluation: Solve (via $Q^{\pi^k}(s, a, h) = r(s, a) + \mathbb{E}_{s' \sim F}$ 2. Policy Improvement: set π_{h}^{k+1}

dynamic programming):

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$$P(\cdot|s,a) = \arg \max_{a} Q^{\pi^{k}}(s, a, h)$$

Again: what if we're in full RL setting where we can't just evaluate expectations? This breaks the Policy Evaluation step, so can we do a fitted version?



- Initialization: choose a policy $\pi^0: S \mapsto A$ • For k = 0, 1, ...1. Policy Evaluation: Solve (via $Q^{\pi^k}(s, a, h) = r(s, a) + \mathbb{E}_{s' \sim P}$ 2. Policy Improvement: set π_{h}^{k+1}
- This breaks the Policy Evaluation step, so can we do a fitted version? Yes! RHS can be written as $\mathbb{E} [r(s_h, r(s_h, r($

dynamic programming):

$$P(\cdot|s,a) \begin{bmatrix} Q^{\pi^{k}}(s', \pi^{k}(a), h+1) \end{bmatrix} \quad \forall s, a, h \\ \forall s, a, h \\ \exists s, a, h \end{bmatrix}$$

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$$a_h$$
) + $Q^{\pi^k}(s_{h+1}, \pi^k(a_h), h+1) | s_h, a_h, h$



- Initialization: choose a policy π⁰ : S → A
 For k = 0,1,...
 1. Policy Evaluation: Solve (via dynamic Q^{π^k}(s, a, h) = r(s, a) + E_{s'~P(·|s,a)} [Q
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dynamic programming):

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$$P(\cdot|s,a) = \arg \max Q^{\pi^{k}}(s, a, h)$$

Again: what if we're in full RL setting where we can't just evaluate expectations?

This breaks the Policy Evaluation step, so can we do a fitted version?

$$a_h) + Q^{\pi^k}(s_{h+1}, \pi^k(\mathbf{s}_h), h+1) \left| s_h, a_h, h \right|$$

Spot the difference!



Fitted Policy Evaluation

Use exact same strategy as before: fixed point iteration

Fitted Policy Evaluation

Use exact same strategy as before: fixed point iteration

Input: policy π , dataset $\tau_1, \ldots \tau_N \sim \rho_{\pi}$ 1. Initialize fitted Q^{π} function at f_0 2. For k = 0, 1, ..., K: $f_{k} = \arg\min_{f \in \mathscr{F}} \sum_{i=1}^{N} \sum_{h=1}^{H-1} \left(f(s_{h}^{i}, a_{h}^{i}, h) - \left(r(s_{h}^{i}, a_{h}^{i}) + f_{k-1}(s_{h+1}^{i}, \pi(\mathbf{g}_{h}^{i}), h+1) \right) \right)^{2}$ 3. Return the function f_K as an estimate of Q^{π}

Fitted Policy Iteration:

• Initialization: choose a policy $\pi^0 : S \mapsto A$ and a sample size N • For k = 0, 1, ...1. Fitted Policy Evaluation: Using N sampled trajectories $\tau_1, \ldots \tau_N \sim \rho_{\pi^k}$, obtain approximation $\hat{Q}^{\pi^k} \approx Q^{\pi^k}$ 2. Policy Improvement: set $\pi_h^{k+1}(s) := \arg \max \hat{Q}^{\pi^k}(s, a, h)$



(Another) Fitted Policy Evaluation option

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Using the definition of the Q function, can do a non-iterative fitted policy evaluation $- \left(\varsigma_{1}, a_{h} \right)$ $Q^{\pi}(s, a, h) = \mathbb{E} \left[\sum_{t=h}^{H-1} \right]_{t=h}^{t=h}$

$$r(s_t, a_t) | (s_h, a_h) = (s, a)$$

(Another) Fitted Policy Evaluation option

Using the definition of the Q function, can do a non-iterative fitted policy evaluation $Q^{\pi}(s, a, h) = \mathbb{E} \left[\sum_{\substack{t=h}}^{H-1} \right]$

> Input: policy π , dataset $\tau_1, \ldots \tau_N \sim \rho$ Return: *N H*-1 $\hat{Q}^{\pi} = \arg\min_{f \in \mathscr{F}} \sum_{i=1}^{n} \sum_{h=1}^{n} \sum_{i=1}^{n} \sum_{h=1}^{n}$

$$r(s_t, a_t) | (s_h, a_h) = (s, a)$$

$$\mathcal{O}_{\pi}$$

 $\left(f(s_{h}^{i}, a_{h}^{i}, h) - \sum_{t=h}^{H-1} r(s_{t}^{i}, a_{t}^{i})\right)^{2}$



- Recap
- Neural networks
- Fitted value iteration
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Attendance: bit.ly/3RcTC9T



Summary:

 Neural Networks work well for complex function approximation with big data Incorporating supervised learning into PI and VI makes them RL algorithms!

> Feedback: bit.ly/3RHtlxy



