Reinforcement Learning & Multi-Armed Bandits

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CS/Stat 184: Introduction to Reinforcement Learning
Fall 2023

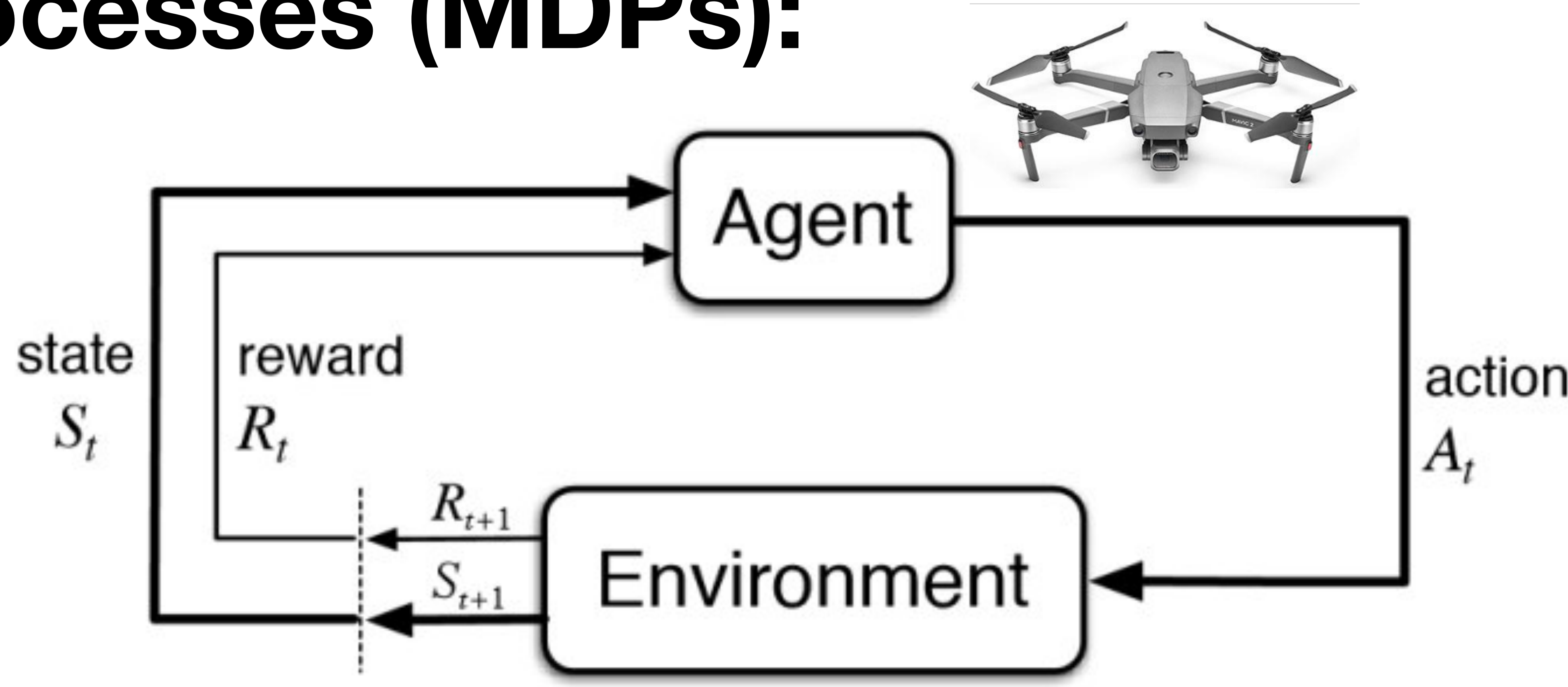
Today

• HW0 due Thurs.

- Recap
- Finite Horizon MDPs
  - Policy Evaluation
  - Optimality
  - The Bellman Equations & Dynamic Programming
- Infinite Horizon MDPs

Recap

Finite Horizon Markov Decision Processes (MDPs):



- An MDP: $\mathscr{M} = \{\mu, S, A, P, r, H\}$
  - μ is a distribution over initial states
    (sometimes we assume we start a given state s_0)
  - S a set of states
  - A a set of actions
  - $P : S \times A \mapsto \Delta(S)$ specifies the dynamics model,
    i.e. $P(s' \mid s, a)$ is the probability of transitioning to s' form states s under action a
  - $r : S \times A \to [0,1]$
    - For now, let's assume this is a deterministic function
    - (sometimes we use a cost $c : S \times A \to [0,1]$)
  - A time horizon $H \in \mathbb{N}$

The Episodic Setting and Trajectories

- Policy $\pi := \{\pi_0, \pi_1, \ldots, \pi_{H-1}\}$
  - deterministic policies: $\pi_t : S \mapsto A$; stochastic policies: $\pi_t : S \mapsto \Delta(A)$
  - we also consider time-dependent policies (but not a function of the history)
- Sampling a trajectory τ on an episode: for a given policy π
  - Sample an initial state $s_0 \sim \mu$:
  - For $t = 0,1,2,\ldots H-1$
    - Take action $a_t \sim \pi_t(\cdot \mid s_t)$ ← choose a_t with pr. $\pi(a_t \mid s_t)$
    - Observe reward $r_t = r(s_t, a_t)$
    - Transition to (and observe) s_{t+1} where $s_{t+1} \sim P(\cdot \mid s_t, a_t)$
  - The sampled trajectory is $\tau = \{s_0, a_0, r_0, s_1, a_1, r_1, \ldots, s_{H-1}, a_{H-1}, r_{H-1}\}$

The Probability of a Trajectory & The Objective

- Probability of trajectory: let $\rho_{\pi,\mu}(\tau)$ denote the probability of observing trajectory $\tau = \{s_0, a_0, r_0, s_1, a_1, r_1, \ldots, s_{H-1}, a_{H-1}, r_{H-1}\}$ when acting under π with $s_0 \sim \mu$.
  - Shorthand: we sometimes write ρ or ρ_π when π and/or μ are clear from context.
  - The rewards in this trajectory must be $r_t = r(s_t, a_t)$ (else $\rho_\pi(\tau) = 0$).
  - For π stochastic:
    $\rho_\pi(\tau) = \mu(s_0)\pi(a_0 | s_0)P(s_1 | s_0, a_0)\ldots\pi(a_{H-2} | s_{H-2})P(s_{H-1} | s_{H-2}, a_{H-2})\pi(a_{H-1} | s_{H-1})$
  - For π deterministic:
    $\rho_\pi(\tau) = \mu(s_0)\mathbf{1}\left(a_0 = \pi(s_0)\right)P(s_1 | s_0, a_0)\ldots P(s_{H-1} | s_{H-2}, a_{H-2})\mathbf{1}\left(a_{H-1} = \pi(s_{H-1})\right)$

- Objective: find policy π that maximizes our expected cumulative episodic reward:

$$\max_{\pi} \mathbb{E}_{\tau \sim \rho_\pi}\left[r(s_0, a_0) + r(s_1, a_1) + \ldots + r(s_{H-1}, a_{H-1})\right]$$

Value function and Q functions:

Quantities that allow us to reason policy's long-term effect:

- Value function $V_h^\pi(s) = \mathbb{E}\left[\sum_{t=h}^{H-1} r(s_t, a_t) \middle| s_h = s\right]$

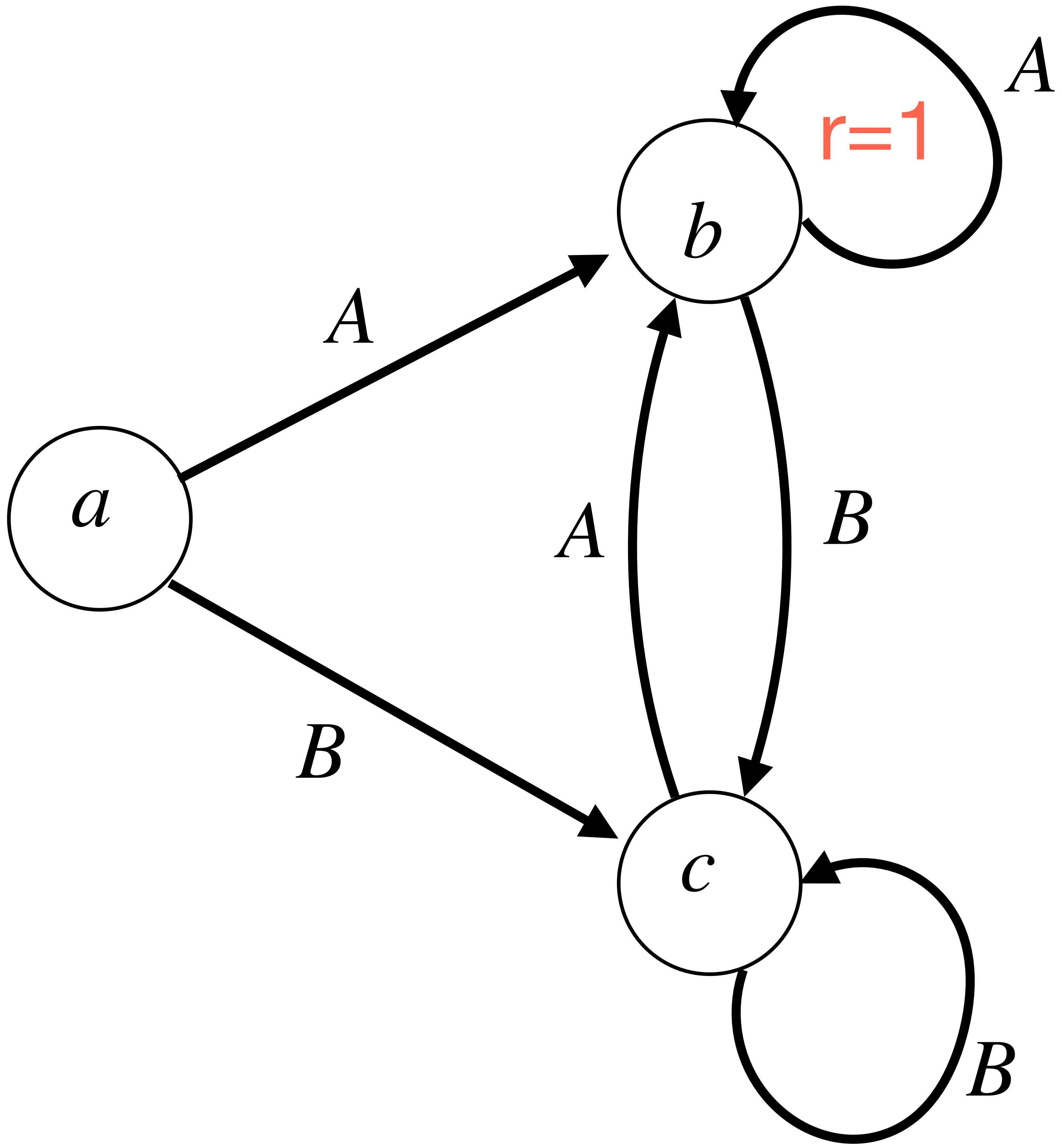
- Q function $Q_h^\pi(s, a) = \mathbb{E}\left[\sum_{t=h}^{H-1} r(s_t, a_t) \middle| (s_h, a_h) = (s, a)\right]$

- At the last stage, for a stochastic policy,:

$$Q_{H-1}^\pi(s, a) = r(s, a) \qquad V_{H-1}^\pi(s) = \sum_a \pi_{H-1}(a \mid s) r(s, a)$$

Example of Policy Evaluation (e.g. computing V^π and Q^π)

Consider the following **deterministic** MDP w/ 3 states & 2 actions, with $H = 3$



Reward: $r(b, A) = 1$, & 0 everywhere else

- Consider the deterministic policy
  $\pi_0(s) = A, \pi_1(s) = A, \pi_2(s) = B, \forall s$

- What is V^π?
  $V_2^\pi(a) = 0, V_2^\pi(b) = 0, V_2^\pi(c) = 0$

  $V_1^\pi(a) = 0, V_1^\pi(b) = 1, V_1^\pi(c) = 0$

  $V_0^\pi(a) = 1, V_0^\pi(b) = 2, V_0^\pi(c) = 1$

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compute $\mathbb{E}_{s_0 \sim \mu}\left[V_0^{\pi}(s_0) \right]$

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  - $V_h^\pi(s) = r(s, \pi_h(s)) + \mathbb{E}_{s' \sim P(\cdot|s,\pi_h(s))}\left[V_{h+1}^\pi(s')\right]$

$\sum_{s'} P(s'|s,\pi_h(s)) V_{h+1}^{\pi}(s')$

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$$= r(s,a) + \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[Q_{h+1}^\pi(s', \pi_{h+1}(s'))\right]$$

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- We use the notation:
$$E_{s' \sim P(\cdot \mid s,a)}[f(s')] = \sum_{s' \in S} P(s' \mid s, a) f(s')$$

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Let r_h denote the random variables $r_h = r(s_h, a_h)$

$$V_h^\pi(s) = E\left[r(s_h, a_h) + r(s_{h+1}, a_{h+1}) + \dots r(s_{H-1}, a_{H-1}) \right]$$

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$\Pr(X, Y)$

$E[X] = E[E[X|Y]]$

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  (assume scalar $+$, $-$, $\times$, $\div$ are $O(1)$ operations) $O(S^2)$

Computation of V^π via Backward Induction

Suppose all we wanted was $V_0^\pi(s_0)$

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- What is the total computational complexity of DP?

$O(S^2 H)$

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Example of Optimal Policy $\pi^\star$

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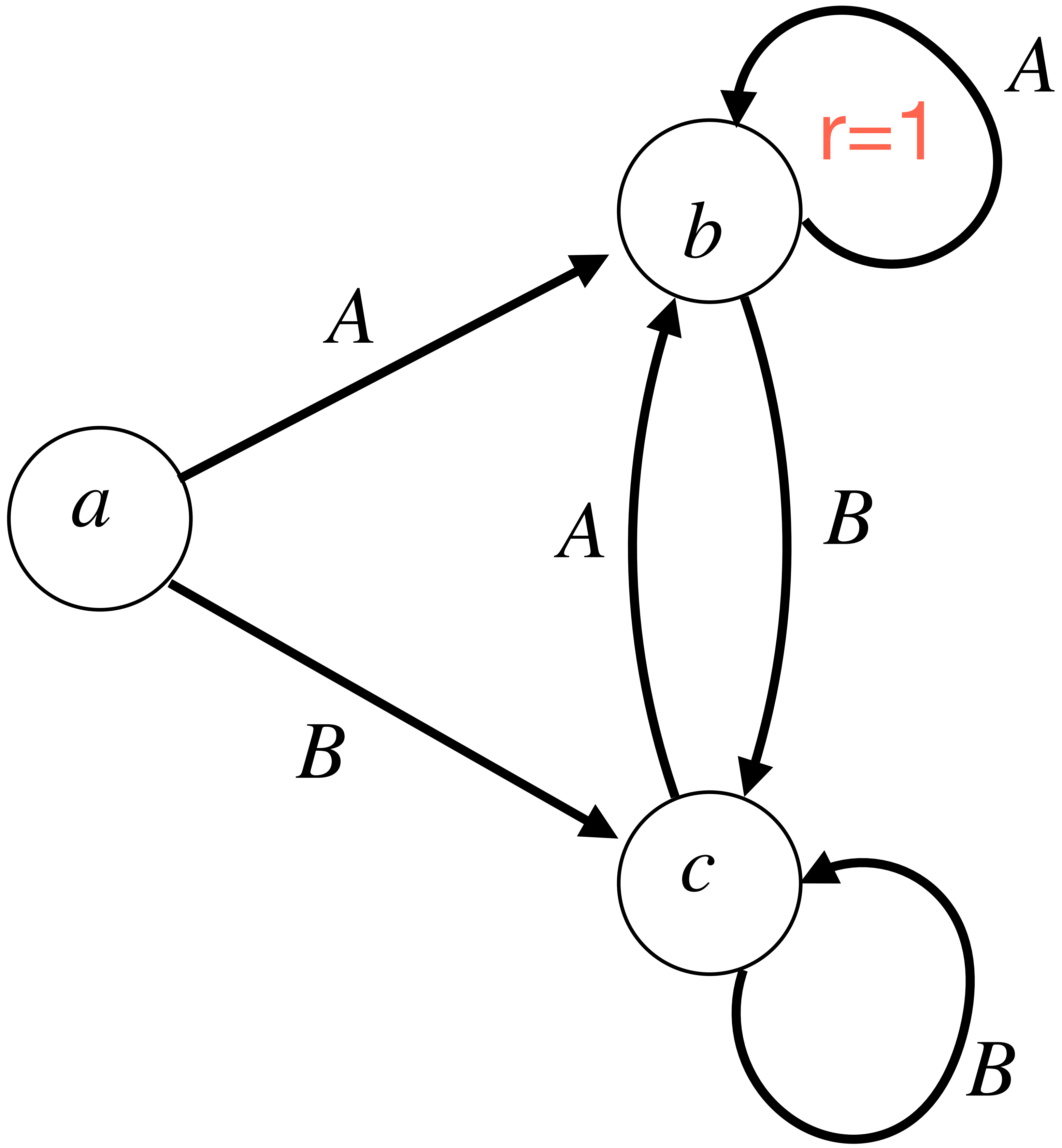

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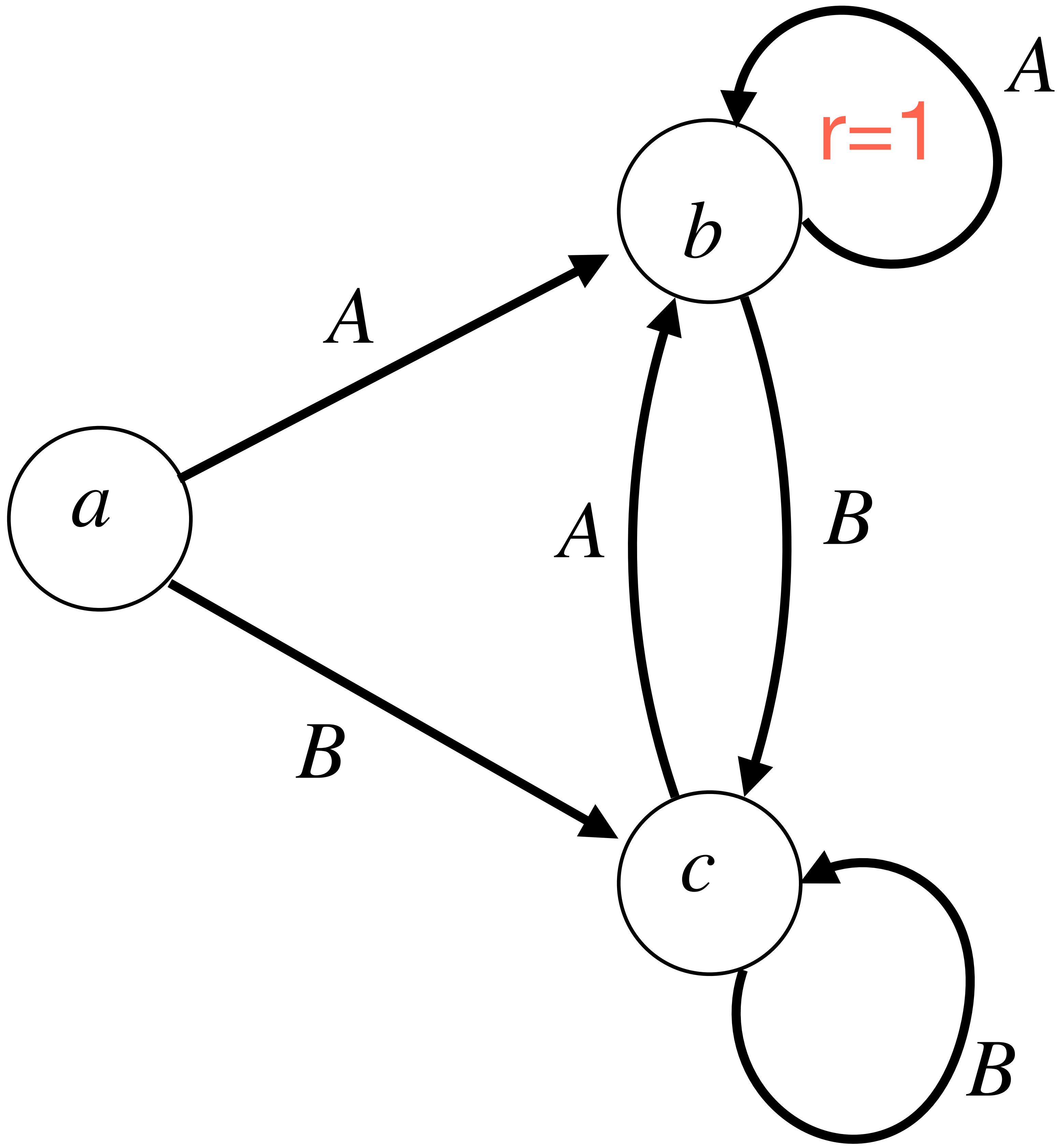

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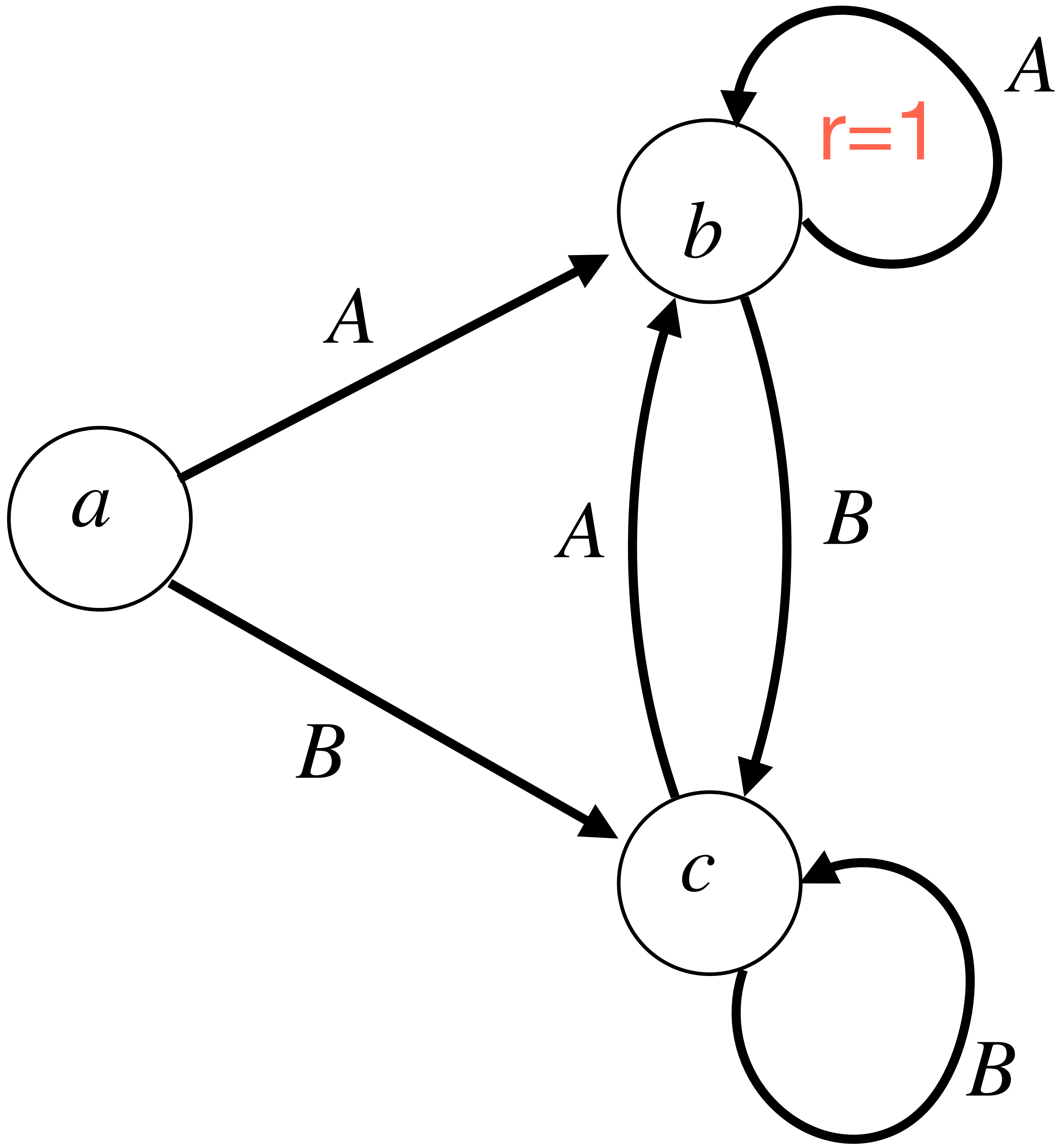

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- What is optimal value function, $V^{\pi^\star} = V^\star$?

  $V_2^\star(a) = 0, V_2^\star(b) = 1, V_2^\star(c) = 0$

  $V_1^\star(a) = 1, V_1^\star(b) = 2, V_1^\star(c) = 1$

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  How many different polices there are?

$\pi_h(s) = a$

$|A|^{|S| H}$

$|A| = 2$

	s: 1, 2, ..., 10
$\pi^{(1)}$	a, a, ..., a
$\pi^{(2)}$	a, S, a, ..., a
⋮	

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- Can we do better?

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- $\implies$ the starting distribution μ doesn't determine $\pi^\star$.

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- A function $V = \{V_0, \ldots V_{H-1}\},\ V_h : S \to R$ satisfies the Bellman equations if

$$V_h(s) = \max_a \left\{ r(s,a) + \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[V_{h+1}(s') \right] \right\}, \forall s$$

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- The optimal policy is: $\pi_h^\star(s) = \arg\max_a \left\{ r(s,a) + \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[V_{h+1}^\star(s')\right] \right\}.$

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- What is the total computational complexity of DP?

$O(S^2 A\ H)$

Summary:

- **Dynamic Programming lets us efficiently compute optimal policies.**
  - We remember the results on “sub-problems”
  - Optimal policies are history independent.

Attendance:
bit.ly/3RcTC9T

Feedback:
bit.ly/3RHtlxy

Today

- Recap
- Finite Horizon MDPs
    - Policy Evaluation
    - Optimality
    - The Bellman Equations & Dynamic Programming
- ✓ Infinite Horizon MDPs

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- Objective: find policy π that maximizes our expected, discounted future reward:

$$\max_{\pi} \mathbb{E}\left[r(s_0, a_0) + \gamma r(s_1, a_1) + \gamma^2 r(s_2, a_2) + \ldots\ldots \,\middle|\, \pi \right]$$

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- Consider a "stationary" policy $\pi : S \mapsto A$
  - "stationary" means not history or time dependent
- Sampling a trajectory τ on an episode: for a given policy π
  - Sample an initial state $s_0 \sim \mu$:
  - For $t = 0,1,2,\ldots\infty$
    - Take action $a_t = \pi(s_t)$
    - Observe reward $r_t = r(s_t, a_t)$
    - Transition to (and observe) s_{t+1} where $s_{t+1} \sim P(\cdot \mid s_t, a_t)$

    $\tau = \{s_0, a_0, r_0, s_1, a_1, r_1, \ldots, \}$

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- Infinite Horizon MDPs
  - ✓ Policy Evaluation
  - Optimality & the Bellman Equations
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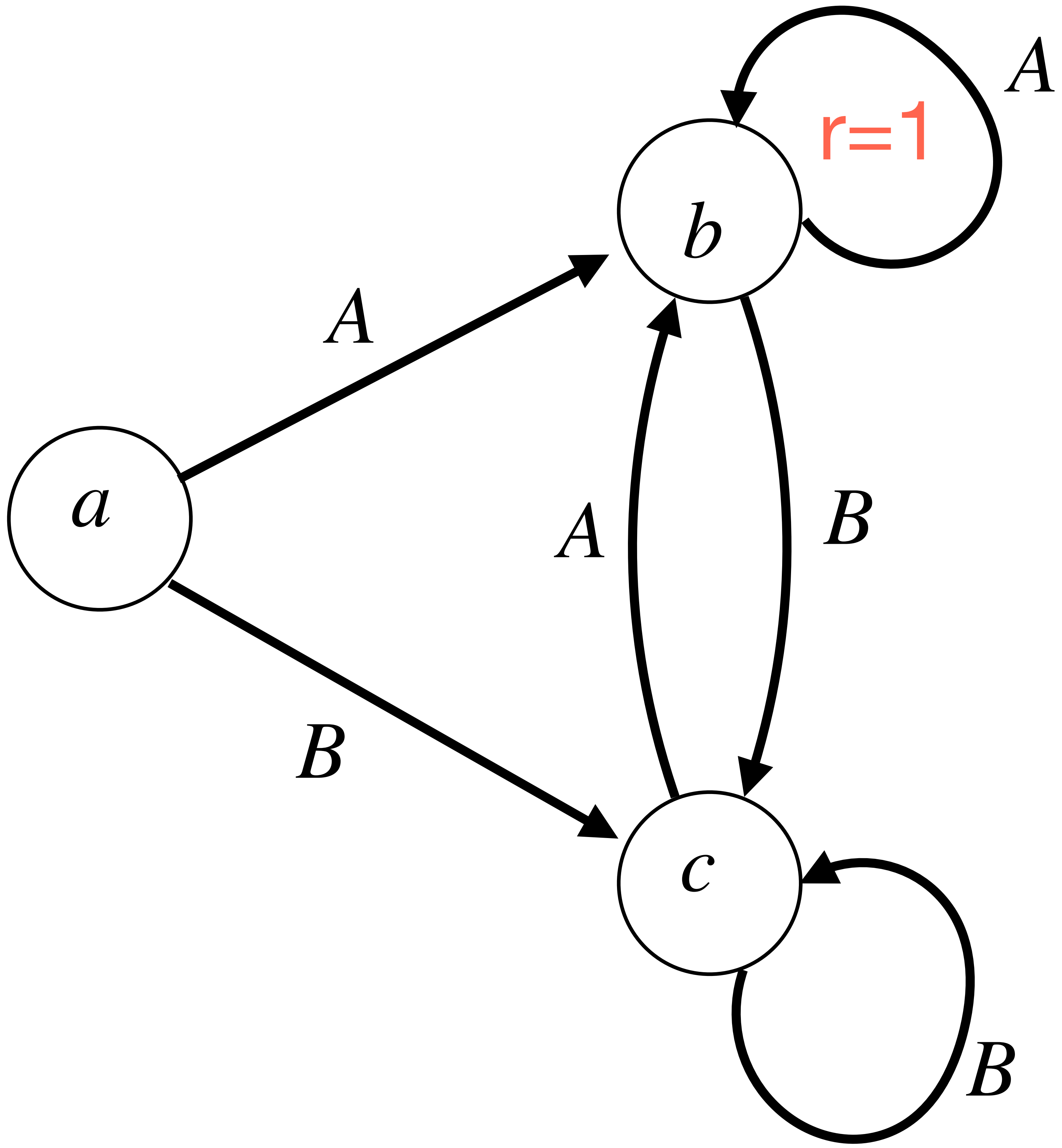
- What are upper and lower bounds on V^π and Q^π

Example of Policy Evaluation (e.g. computing V^π and Q^π)

Consider the following **deterministic** MDP w/ 3 states & 2 actions



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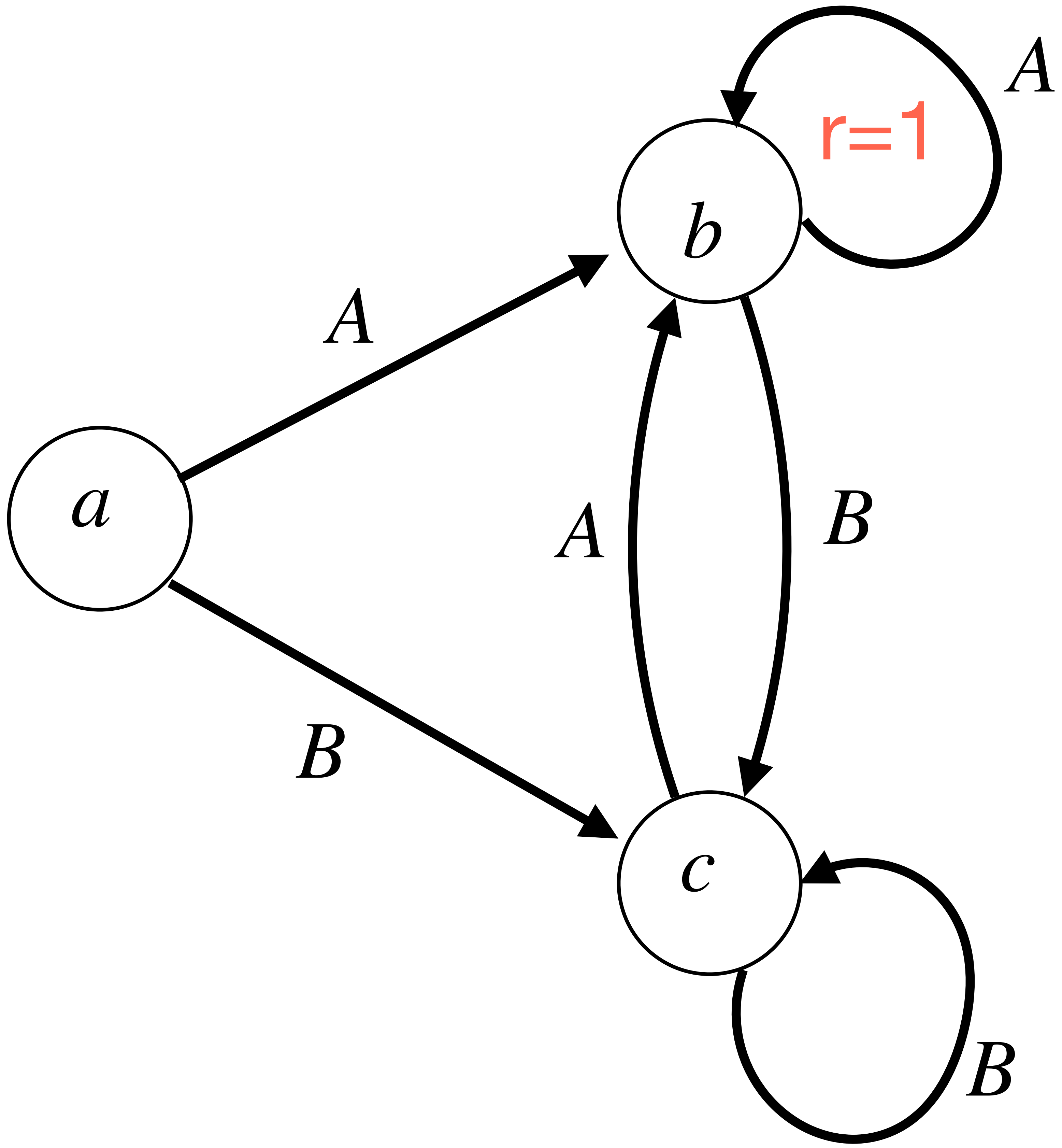

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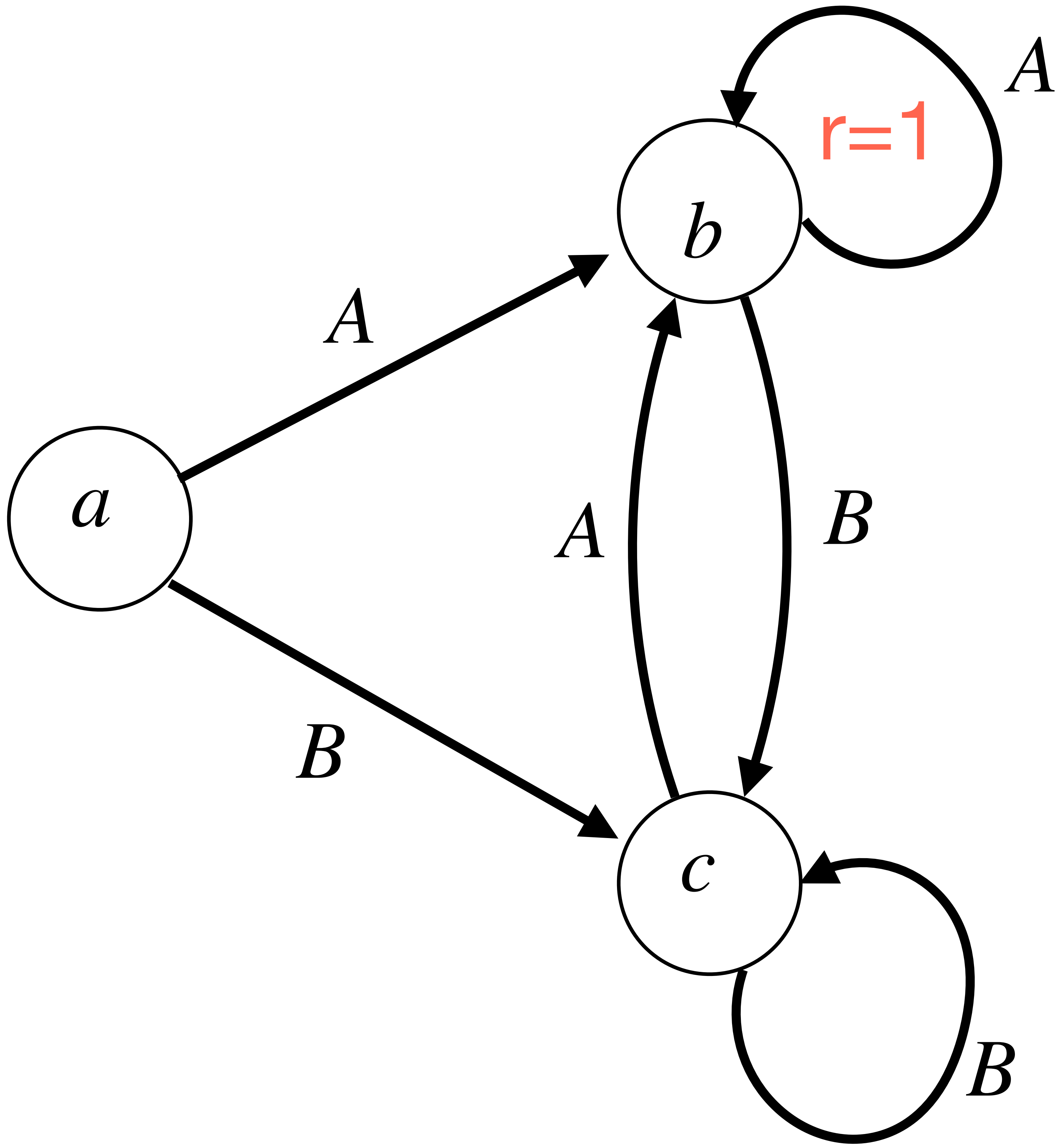

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- What is V^π?
  $V^\pi(a) =$

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