Infinite Horizon MDPs

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CS/Stat 184: Introduction to Reinforcement Learning

Fall 2023

Today

- Recap
- Infinite Horizon MDPs
    - Policy Evaluation
    - Optimality & the Bellman Equations
    - Value Iteration
    - Policy Iteration

Recap

Bellman Consistency (theorem)

- For a fixed policy, $\pi := \{\pi_0, \pi_1, \ldots, \pi_{H-1}\}$, $\pi_h : S \mapsto A, \forall h$,
- By definition, $V_h^\pi(s) = Q_h^\pi(s, \pi_h(s))$
- At $H - 1$, $Q_{H-1}^\pi(s, a) = r(s, a)$, $V_{H-1}^\pi(s) = r(s, \pi_{H-1}(s))$
- Bellman consistency conditions: for a given policy π,
  - $V_h^\pi(s) = r(s, \pi_h(s)) + \mathbb{E}_{s' \sim P(\cdot|s, \pi_h(s))}\left[V_{h+1}^\pi(s')\right]$
  - $Q_h^\pi(s, a) = r(s, a) + \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[V_{h+1}^\pi(s')\right]$

V^π, Q^π satisfy

Computation of V^π via Backward Induction

- For a fixed policy, $\pi := \{\pi_0, \pi_1, \ldots, \pi_{H-1}\}$, $\pi_h : S \mapsto A, \forall h$,

  Bellman consistency $\Longrightarrow$ we can compute V_h^π, assuming we know the MDP.

- Init: $V_H^\pi(s) = 0, \ \forall s \in S$
- For $h = H-1, \ldots 0$, set:

  $V_h^\pi(s) = r(s, \pi_h(s)) + \mathbb{E}_{s' \sim P(\cdot|s,\pi_h(s))} \left[V_{h+1}^\pi(s') \right], \ \forall s \in S$

- What is the per iteration computational complexity of DP?

  (assume scalar $+, -, \times, \div$ are $O(1)$ operations)
- What is the total computational complexity of DP?

Properties of an Optimal Policy $\pi^\star$

- Let Π be the set of all time dependent, history dependent, stochastic policies.

- **Theorem:** Every finite horizon MDP has a deterministic, history-independent optimal policy, that dominates all other policies, everywhere.
  - i.e. there exists a policy $\pi^\star := \left\{\pi_0^\star, \pi_1^\star, \ldots, \pi_{H-1}^\star\right\}, \pi_h^\star : S \mapsto A$ such that

$$V_h^{\pi^\star}(s) \geq V_h^{\pi}(s) \quad \forall s, h, \forall \pi \in \Pi$$

The Bellman Equations

- A function $V = \{V_0, \ldots V_{H-1}\},\ V_h : S \to R$ satisfies the Bellman equations if

$$V_h(s) = \max_a \left\{ r(s,a) + \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[V_{h+1}(s') \right] \right\}, \forall s, h$$

  (assume $V_H = 0$).

- **Theorem:**
  - V satisfies the Bellman equations if and only if $V = V^\star$.
  - The optimal policy is: $\pi_h^\star(s) = \arg\max_a \left\{ r(s,a) + \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[V_{h+1}^\star(s') \right] \right\}.$

$$Q^{\pi^\star}(s,a) = Q^\star(s,a)$$

Computation of $V^\star$ with Dynamic Programming

space complexity $O(|S|H)$

- Theorem: the following Dynamic Programming algorithm correctly computes $\pi^\star$ and $V^\star$
  Prf: the Bellman equations directly lead to this backwards induction.

> - Initialize: $V^\pi_H(s) = 0 \ \forall s \in S$
>
>   For t= $H-1,\ldots 0$, set:
>
>   - $V^\star_h(s) = \max_a \left[r(s,a) + \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[V^\star_{h+1}(s') \right] \right], \ \forall s \in S$
>   - $\pi^\star_h(s) = \arg\max_a \left[r(s,a) + \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[V^\star_{h+1}(s') \right] \right], \ \forall s \in S$

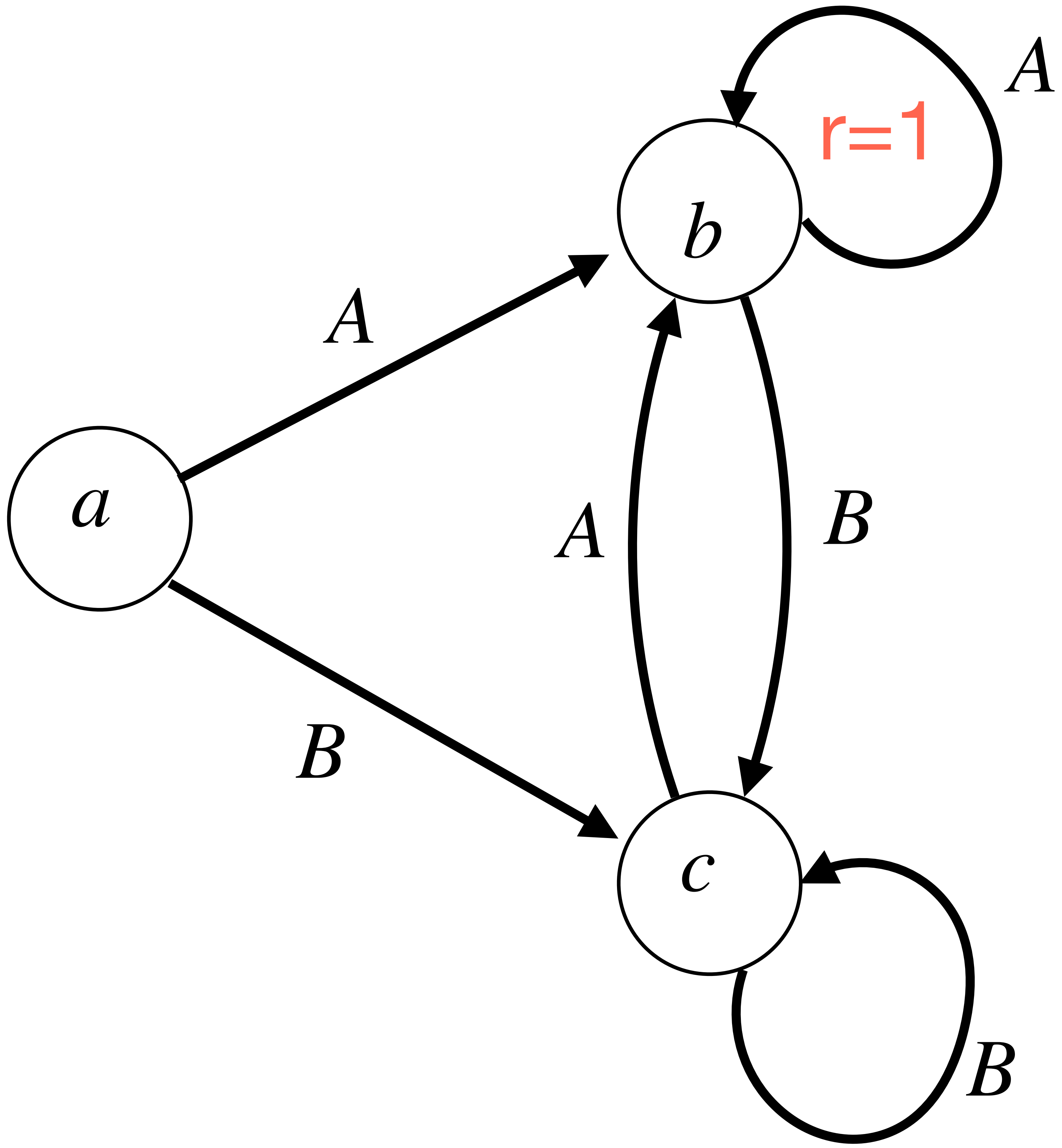
- What is the per iteration computational complexity of DP?
  (assume scalar $+, -, \times, \div$ are $O(1)$ operations)

  $O(|A| \cdot |S|^2)$
- What is the total computational complexity of DP?

  $O(H \cdot$ " $)$

Example of Optimal Policy $\pi^\star$

Consider the following **deterministic** MDP w/ 3 states & 2 actions, with $H = 3$



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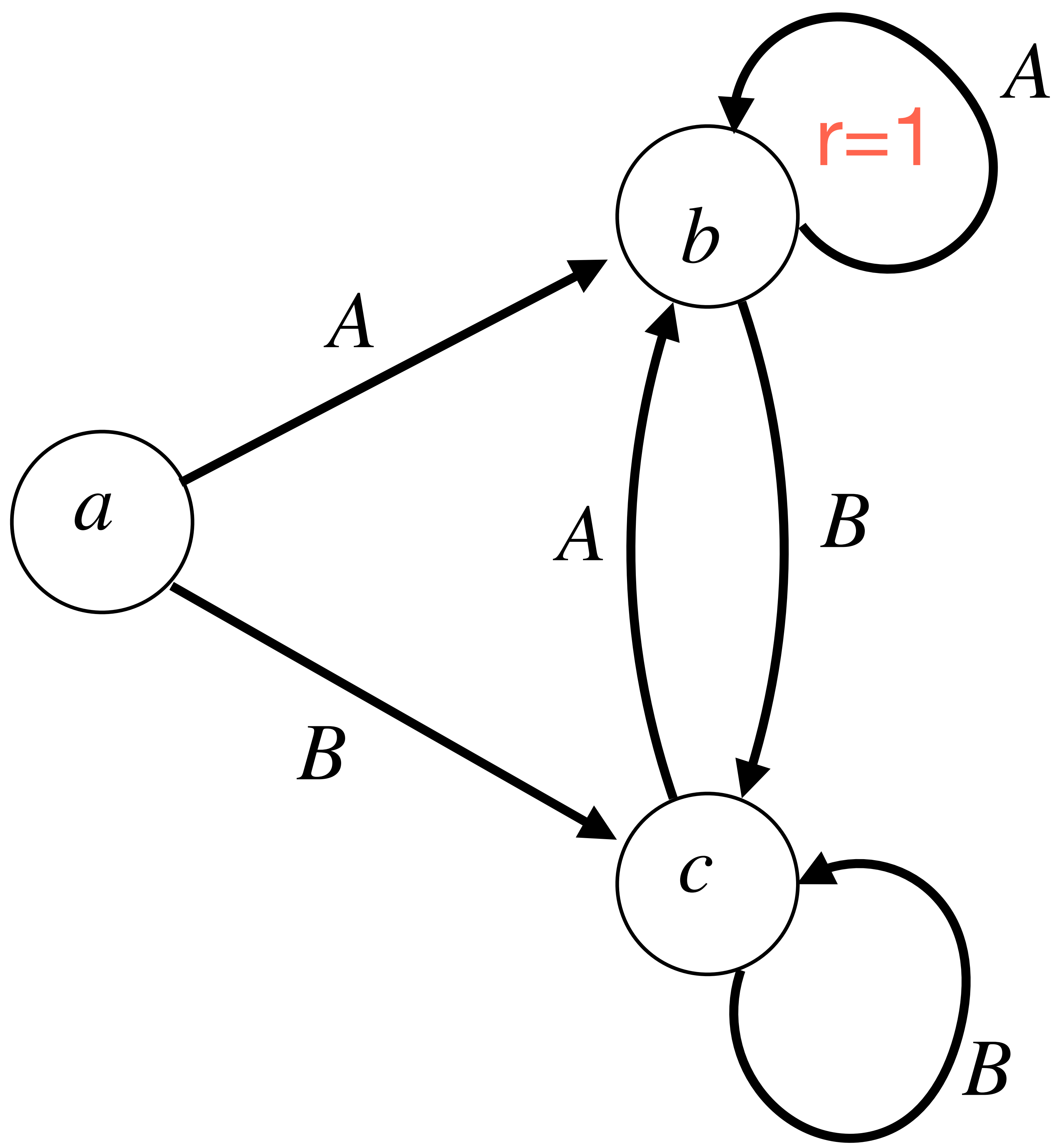

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- What is optimal value function, $V^{\pi^\star} = V^\star$?

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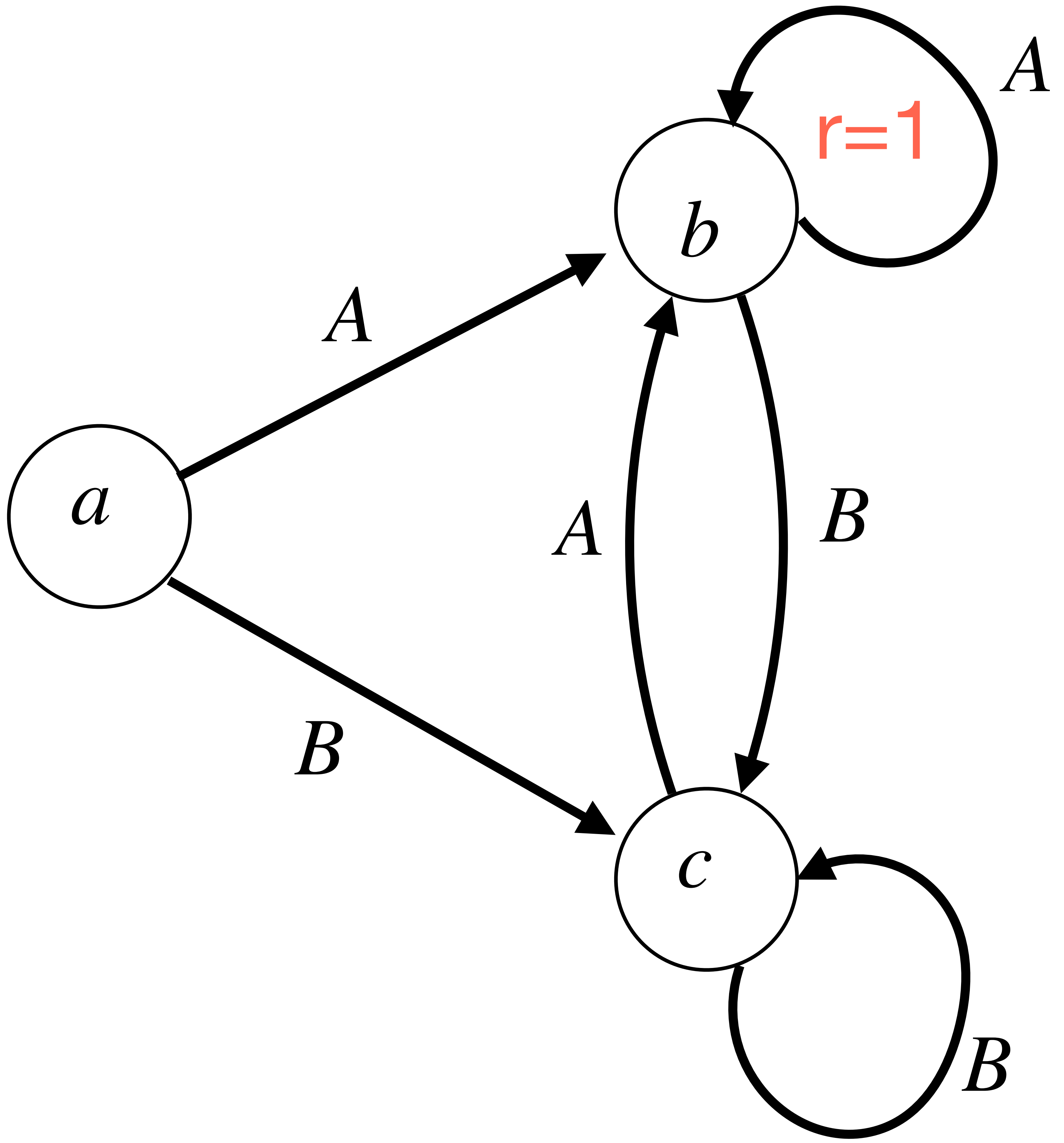

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- What is optimal value function, $V^{\pi^\star} = V^\star$?

$V_2^\star(a) = 0, V_2^\star(b) = 1, V_2^\star(c) = 0$

$V_2^\star(a) = \max_{\{A,B\}} \{0 + V_3(b), 0 + V_3(c)\}$

do this b, c

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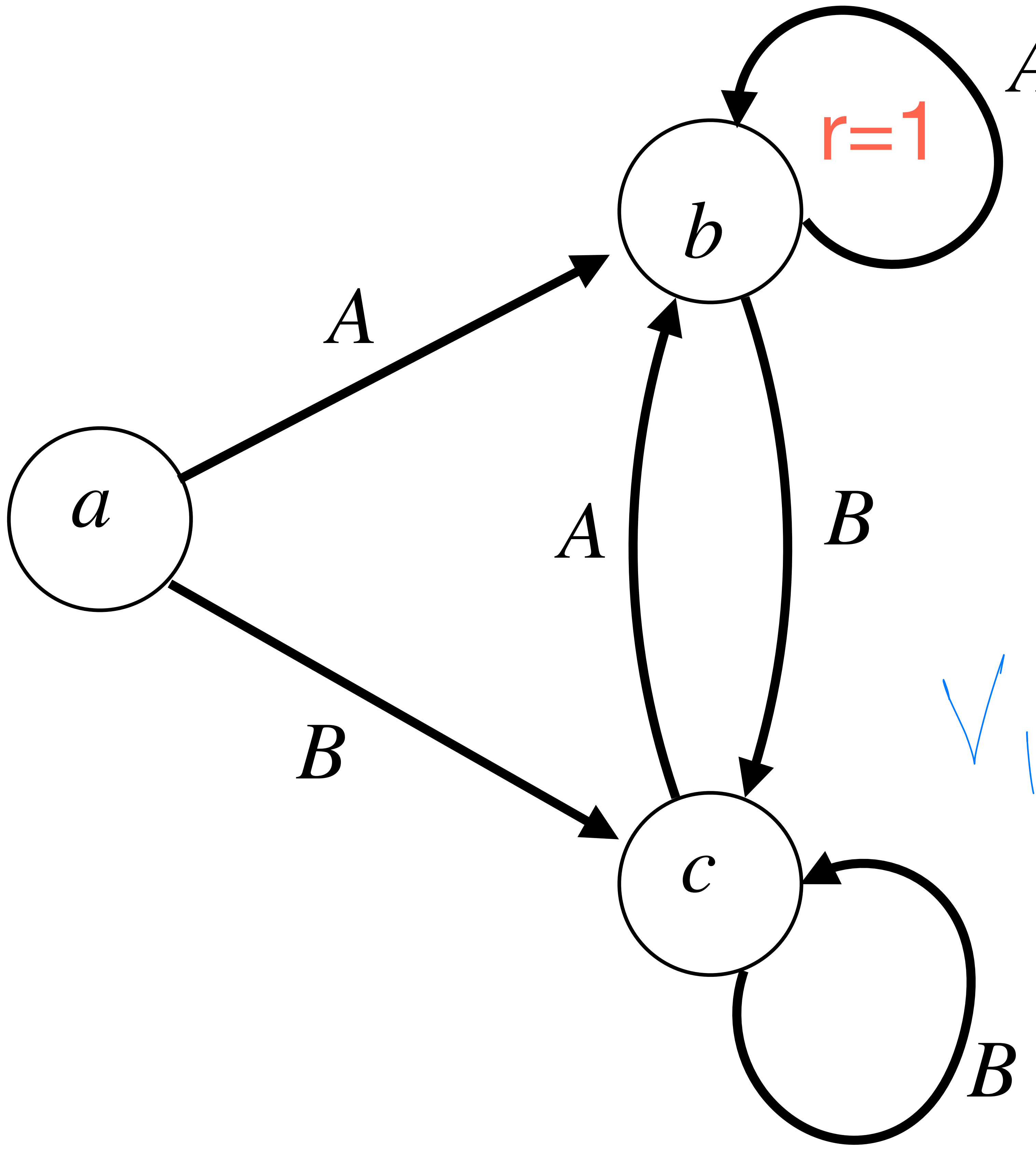

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$V_2^\star(a) = 0, V_2^\star(b) = 1, V_2^\star(c) = 0$

$V_1^\star(a) = 1, V_1^\star(b) = 2, V_1^\star(c) = 1$

$V_1^*(a) = \max_{\{A,B\}}\{0 + V_2^*(b), 0 + V_2^*(c)\} = \max\{1, 0\} = 1$ ✓

$r(a,A)$ 1 0

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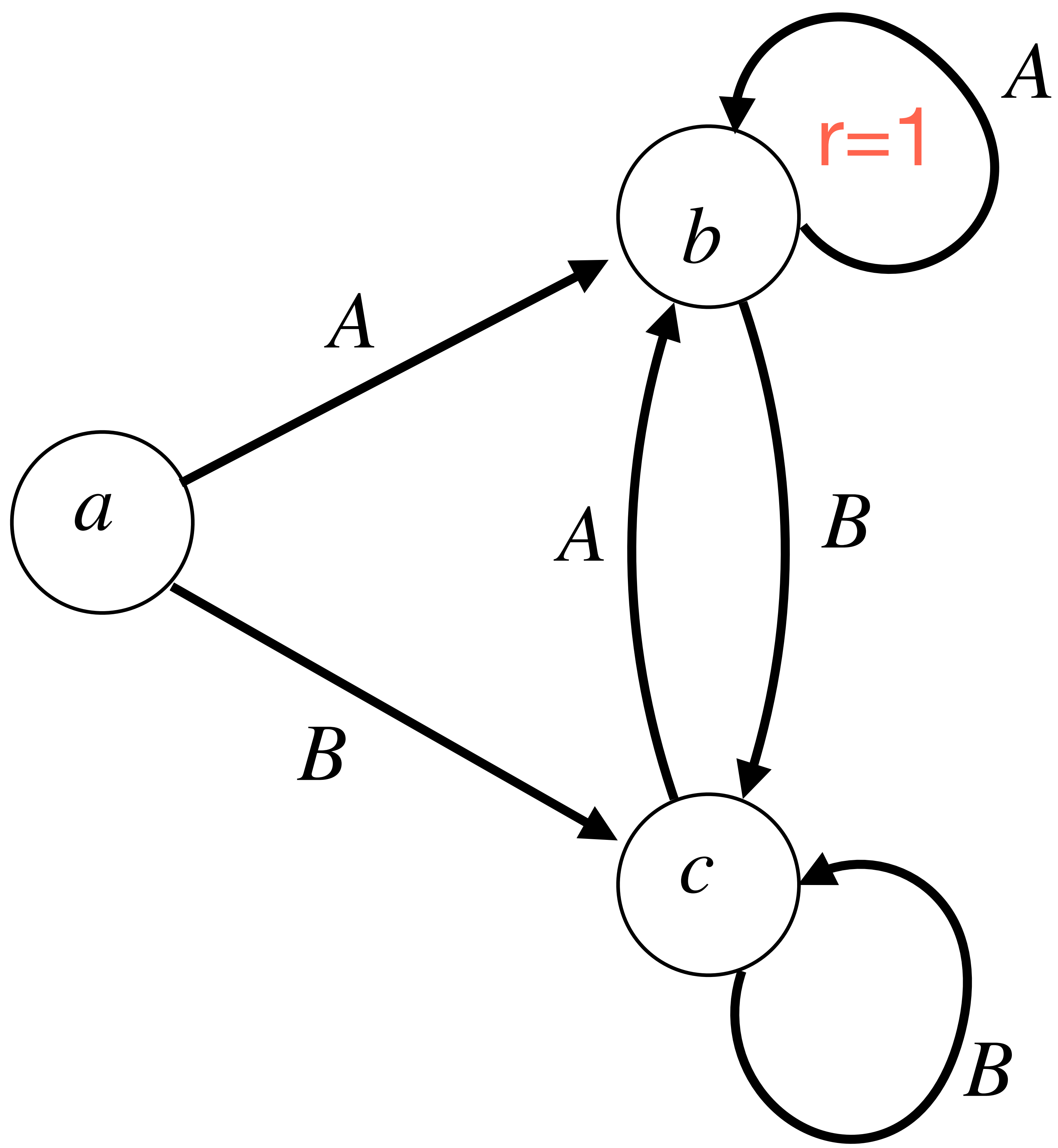

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$V_0^\star(a) = 2, V_0^\star(b) = 3, V_0^\star(c) = 2$

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  - instead of finite horizon H, we have a discount factor $\gamma \in [0,1)$

$r_0 + \gamma r_1 + \gamma^2 r_2 + \gamma^3 r_3 + \ldots$

- Objective: find policy π that maximizes our expected, discounted future reward:

$$\max_{\pi} \mathbb{E}\left[r(s_0, a_0) + \gamma r(s_1, a_1) + \gamma^2 r(s_2, a_2) + \ldots\ldots \,\middle|\, \pi \right]$$

Same as "total reward" in game which ends with Pr $(1-\gamma)$ at every step

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  - For $t = 0,1,2,\ldots\infty$
    - Take action $a_t = \pi(s_t)$
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- The infinite trajectory: $\tau = \{s_0, a_0, r_0, s_1, a_1, r_1, \ldots, \}$

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- What are upper and lower bounds on V^π and Q^π

Example of Policy Evaluation (e.g. computing V^π and Q^π)

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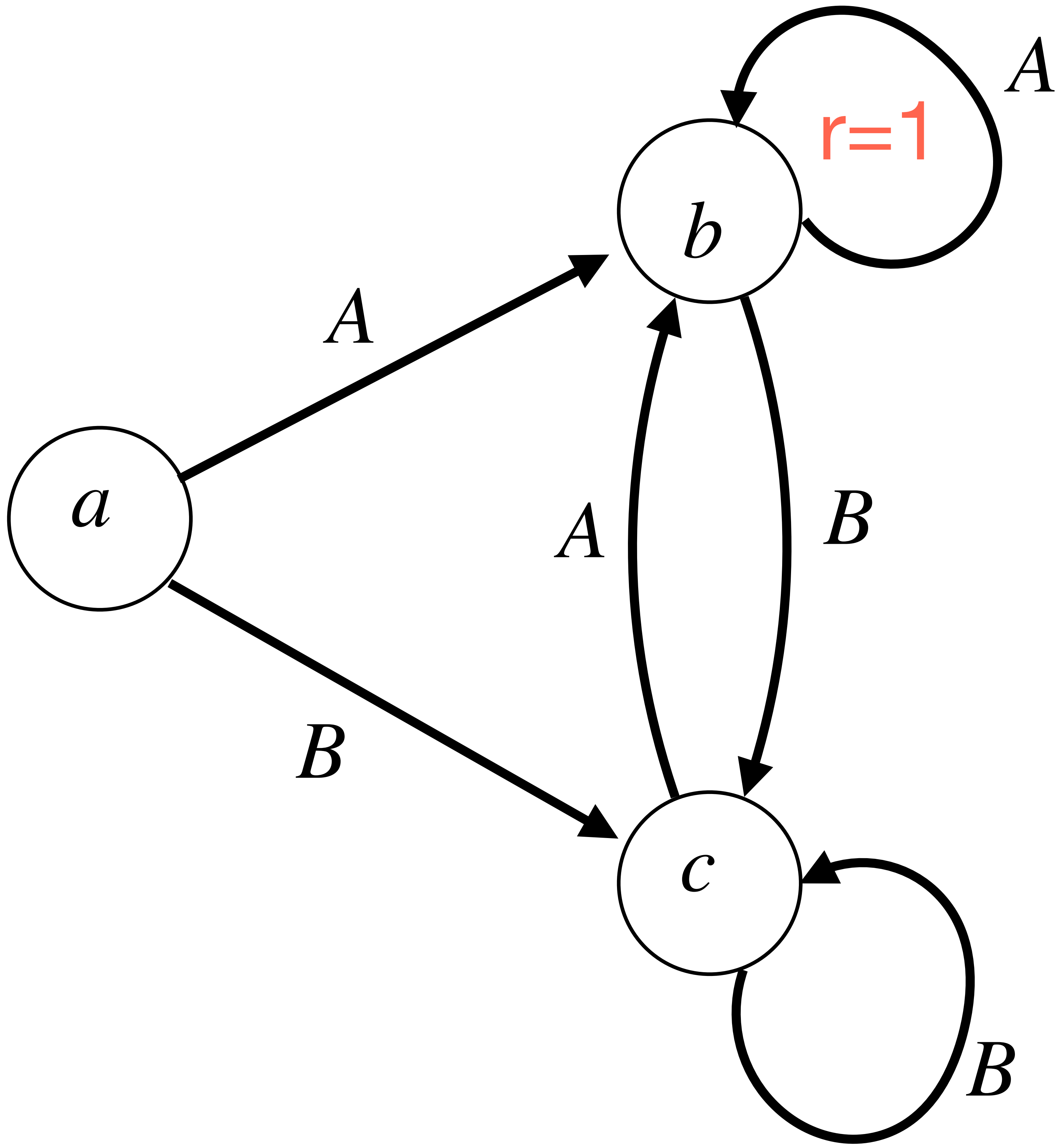

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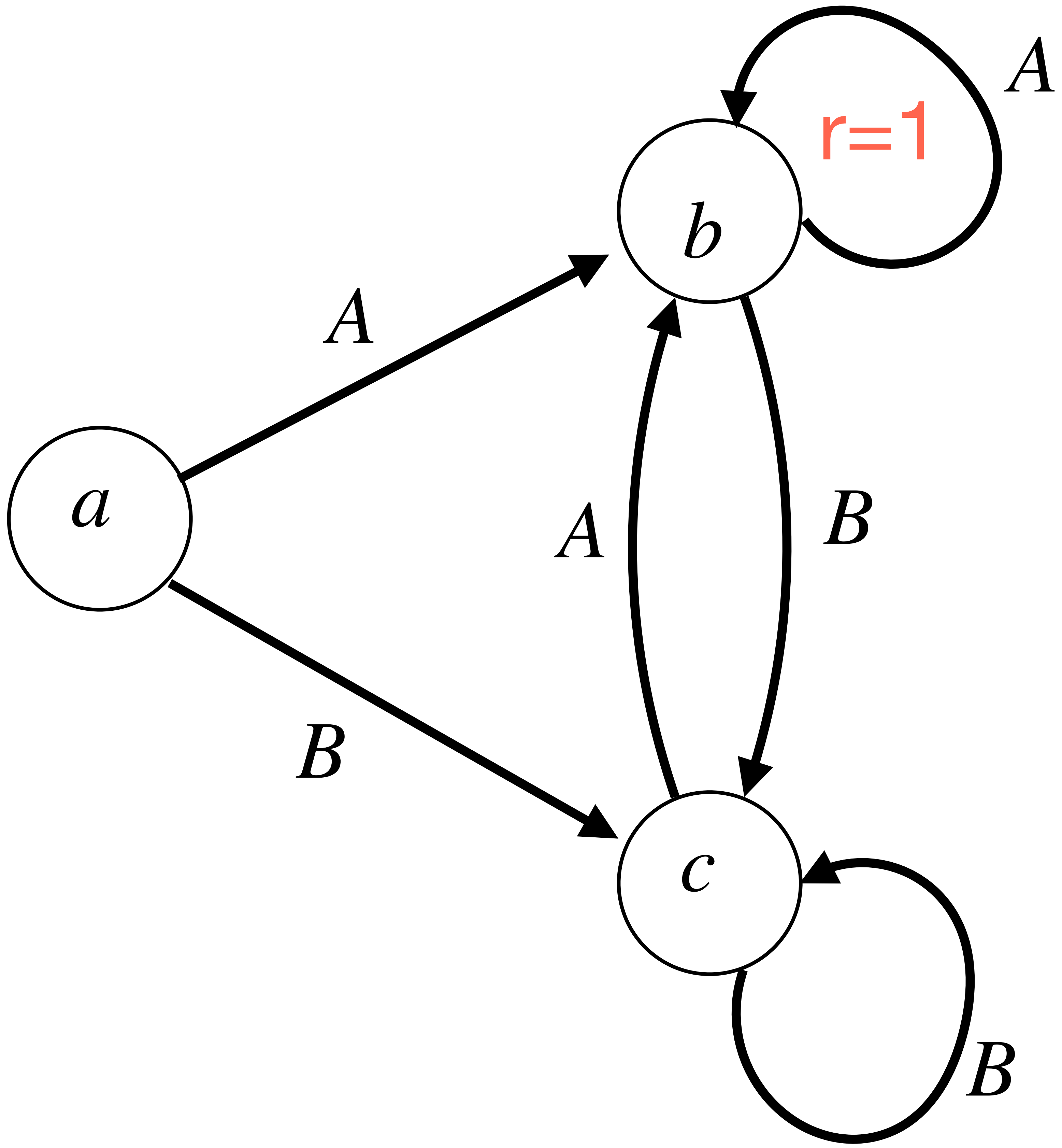

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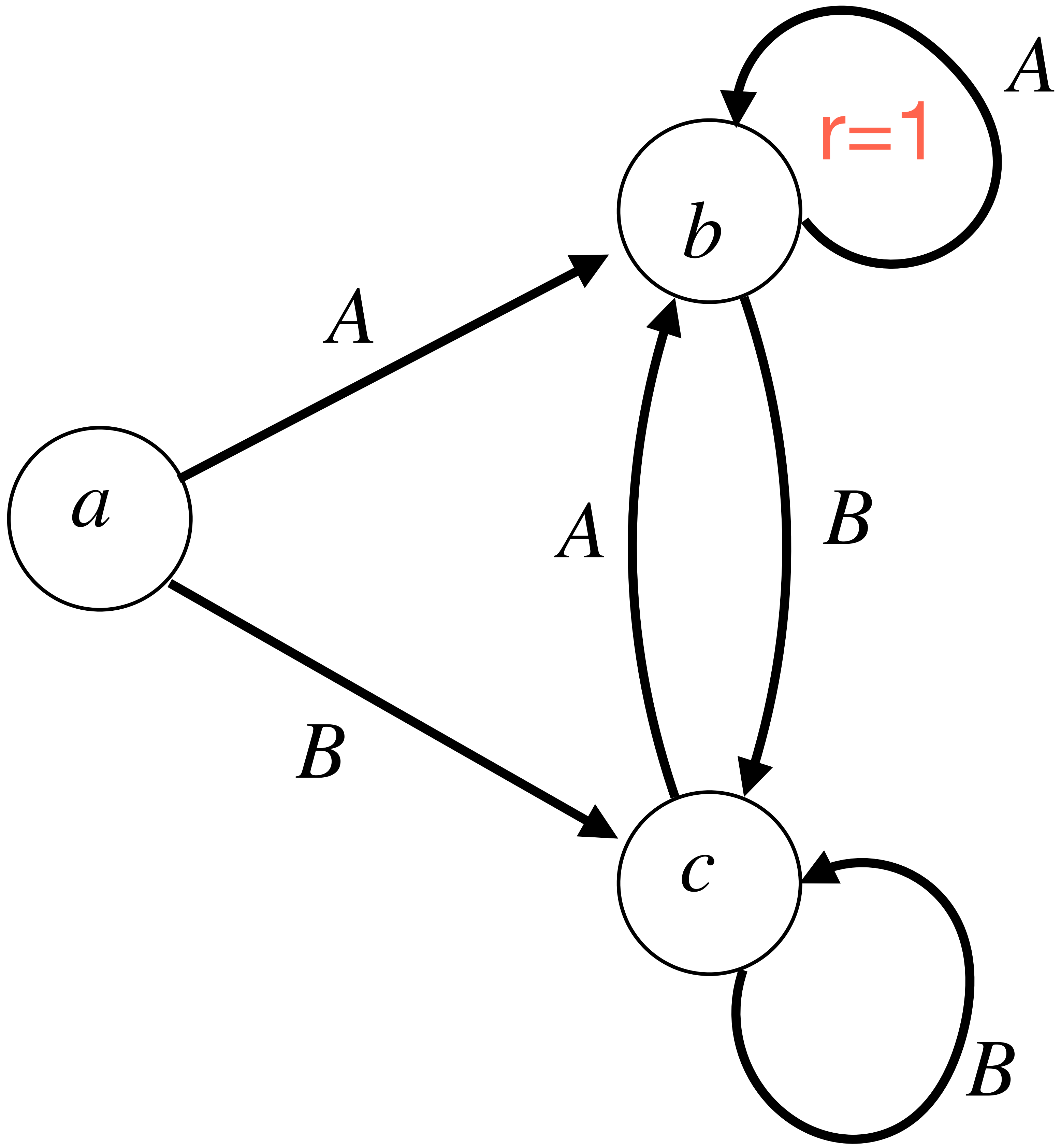

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- What is V^π?

  $V^\pi(a) =$ $0 + \gamma \cdot 0 + \gamma^2 \cdot 1 + \gamma^3 \cdot 1$ $\frac{\gamma^2}{1-\gamma}$

  $V^\pi(b) =$ $\frac{1}{1-\gamma}$

  $V^\pi(c) =$ $0 + \gamma 1 + \gamma^2 1 +$

  $\frac{\gamma}{1-\gamma}$

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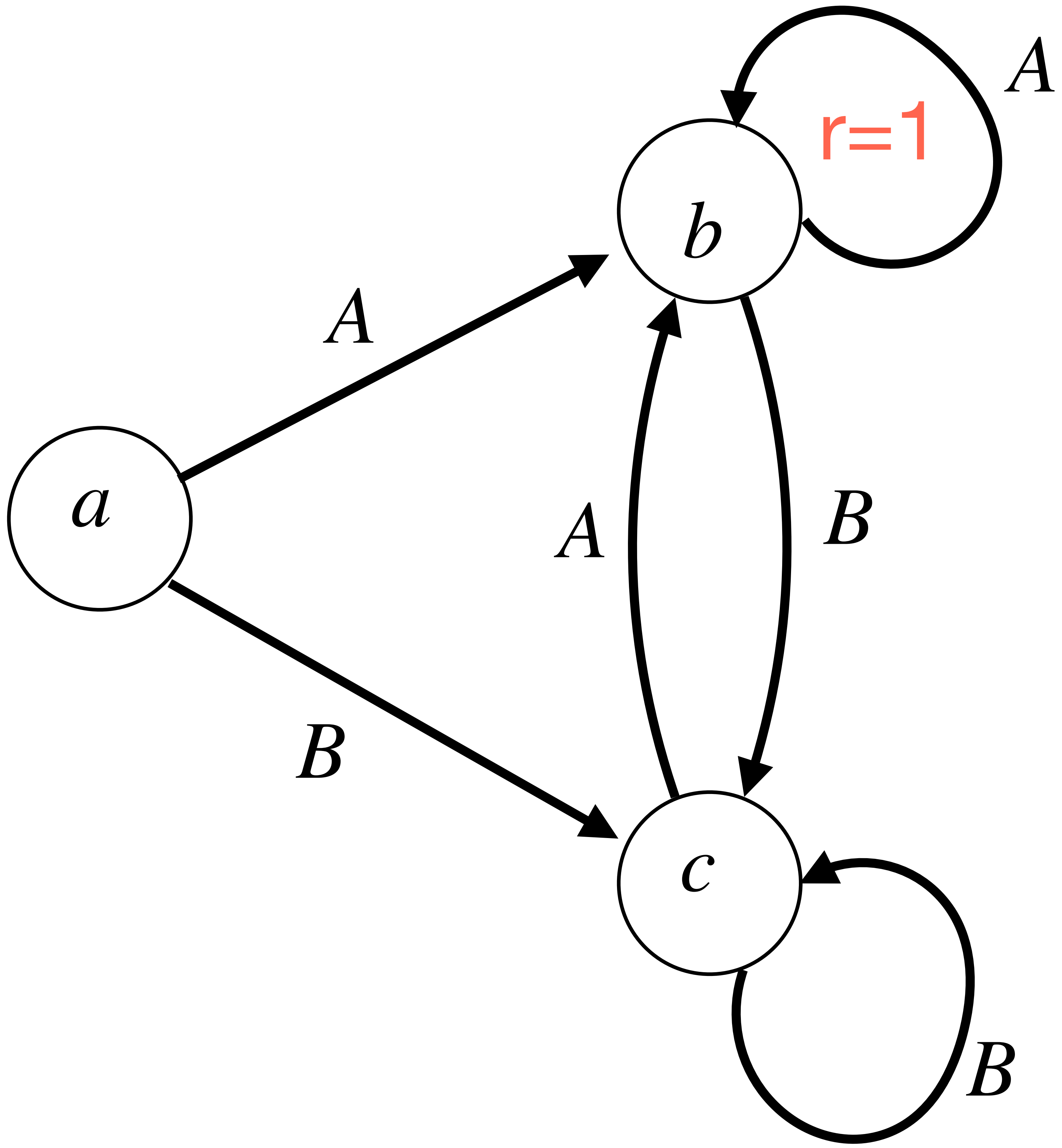

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  $\pi(a) = B, \pi(b) = A, \pi(c) = A$
- What is V^π?
  $V^\pi(a) = \gamma^2/(1-\gamma)$

  $V^\pi(b) = 1/(1-\gamma)$

  $V^\pi(c) = \gamma/(1-\gamma)$

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$\sim r_0(\ ) + \gamma r_1 + \gamma^2 r_2 + \dots$

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- We have the Bellman consistency conditions, for a given policy π

$$V^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s'} P(s' | s, \pi(s)) V^\pi(s')$$

x+y=1
x-y=0

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∀s

find $V^{\pi}(s)$

given γ, P, r

- How do we use this to find a solution?

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- How do we use this to find a solution?
- What is the time complexity?

$O(|S|^3)$

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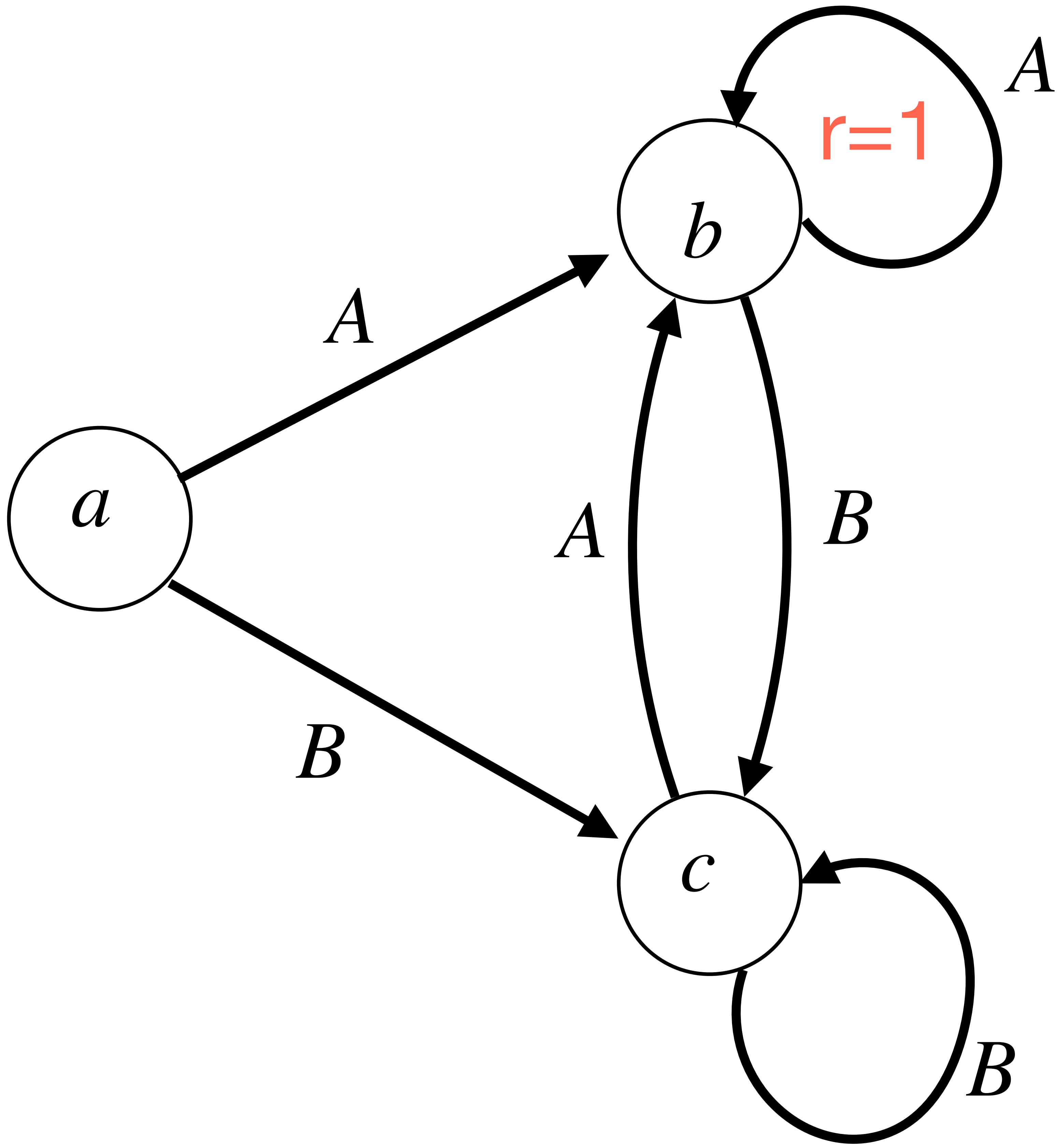

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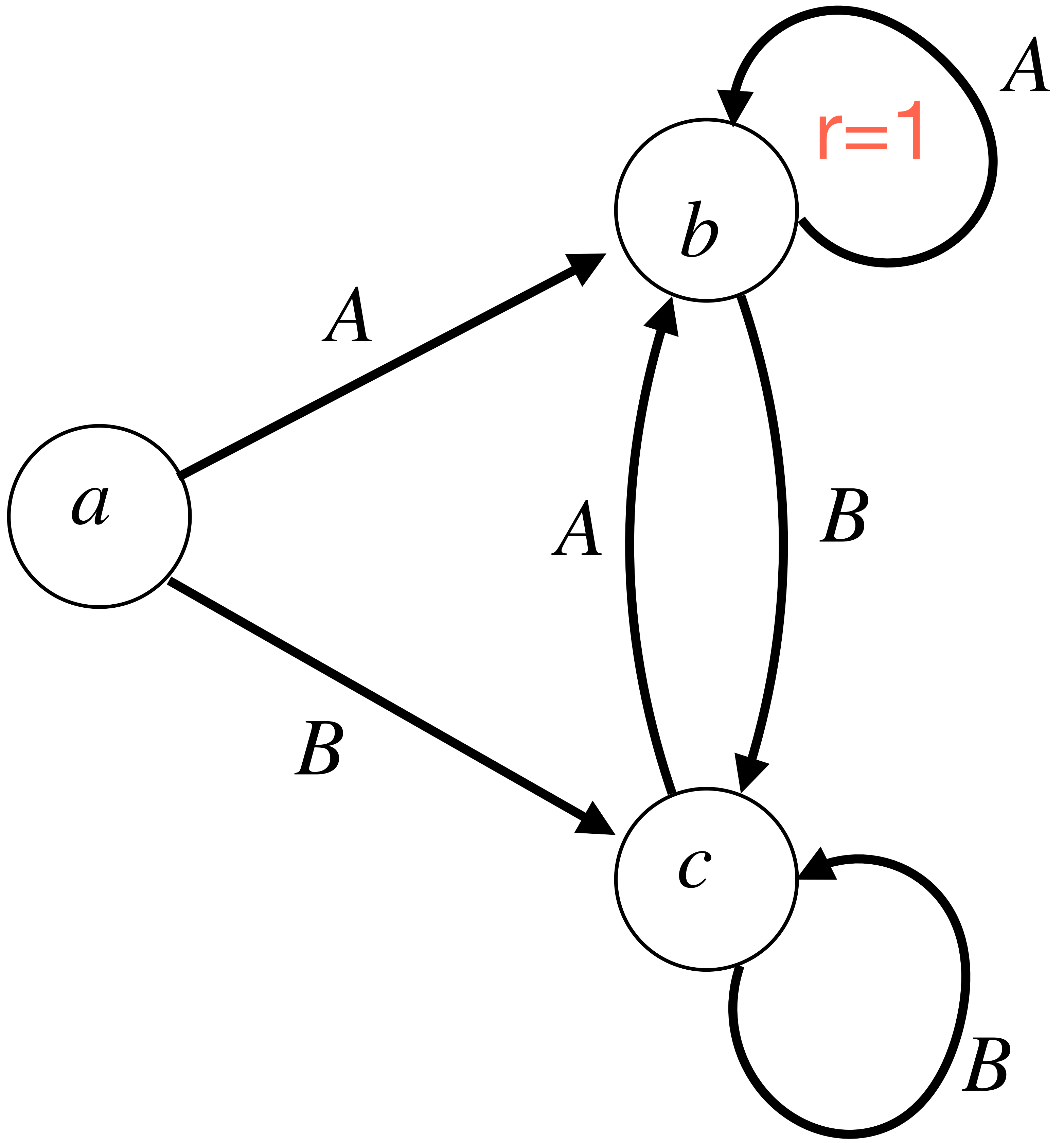

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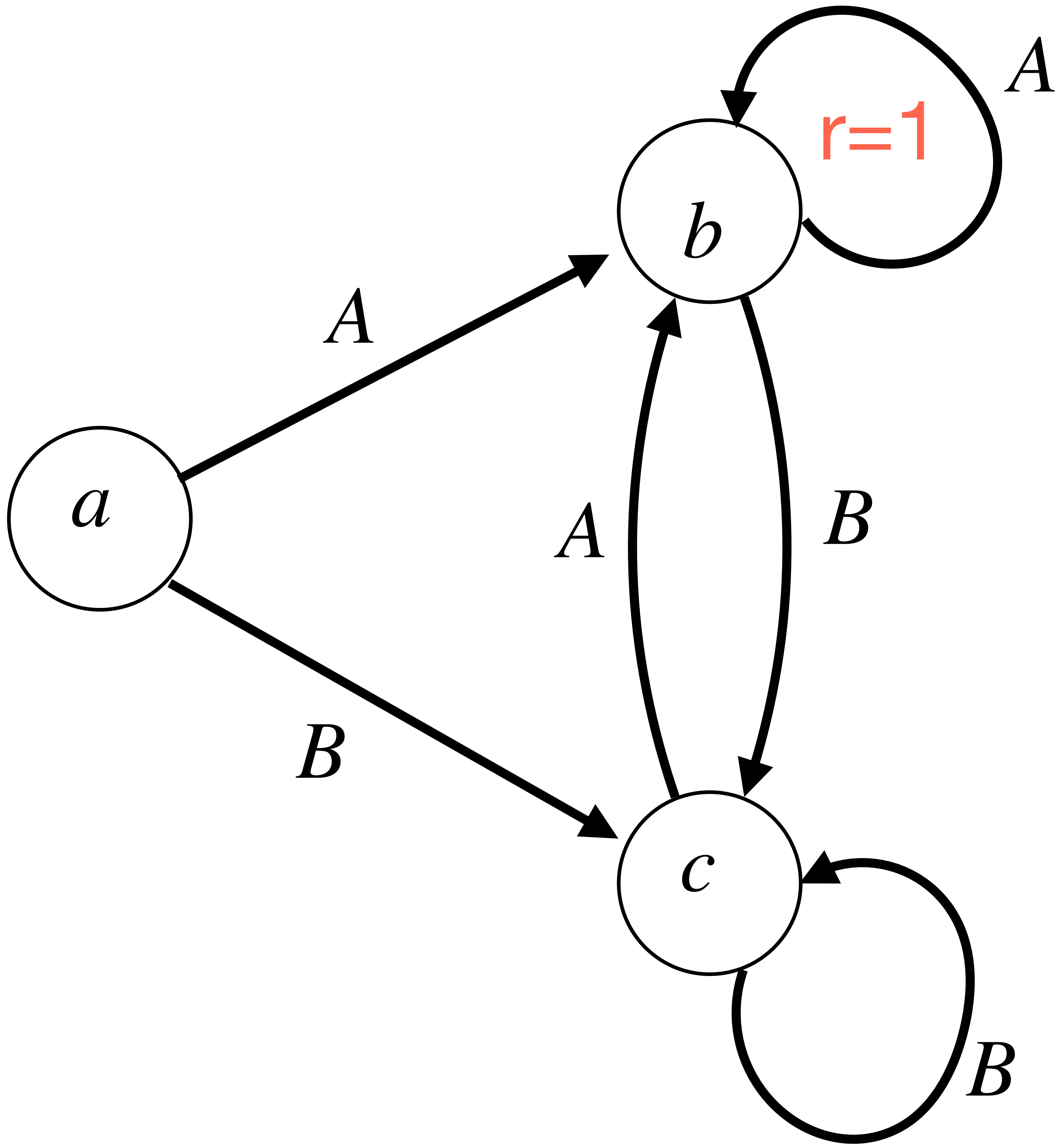

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$$V^\star(a) = \frac{\gamma}{1-\gamma},\ V^\star(b) = \frac{1}{1-\gamma},\ V^\star(c) = \frac{\gamma}{1-\gamma}$$

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- Naively, we could compute the value of all policies and take the best one.
- Suppose $|S|$ states, $|A|$ actions.
  How many different stationary polices are there?

$\pi : S \to A$

det

$|A|^{(S)}$

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- $\implies$ we can write: $V^\star = V^{\pi^\star}$ and $Q^\star = Q^{\pi^\star}$.

Summary:

- **Discounted infinite horizon MDP:**
  - Concepts: Policy Eval; Bellman equations; Value Iteration

Attendance:
bit.ly/3RcTC9T

Feedback:
bit.ly/3RHtlxy